Separable equation / Existence and Uniqueness of Solutions.

Ex (x independent variable

$$\frac{dy}{dx} = -\frac{x}{y}$$ First order non linear

Solve by separation of variables

$$\int y\,dy = \int -x\,dx \Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + c$$

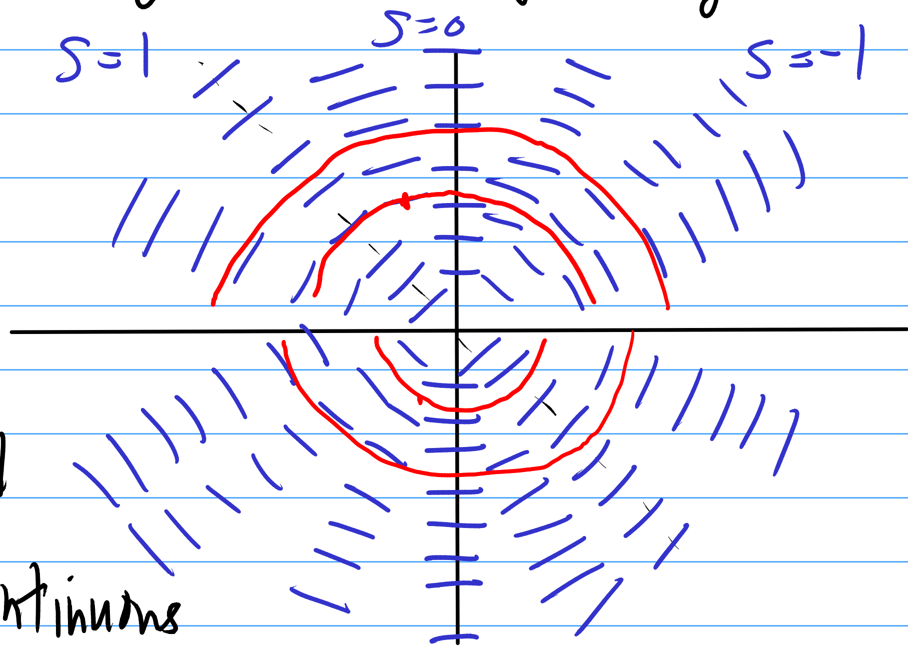
$$\Rightarrow y = \pm\sqrt{-x^2 + 2C} \qquad \Updownarrow \qquad x^2 + y^2 = 2C$$

equation of a circle!

Picture: Direction field and solution curves

$$\frac{dy}{dx} = -\frac{x}{y} \qquad f(x,y) = -\frac{x}{y}$$

where does the dir. field have slope $= S$?



$$-\frac{x}{y} = S$$

$$y = -\frac{1}{S}x$$

set where slope $= S$

$y = 0$

$\Downarrow$

$-\frac{x}{y}$ undefined

$f(x,y)$ not continuous

Observe: I want solutions to be functions of x, but entire circles don't pass vertical line test

→ lower and upper semicircles represent two separate solutions

→ There is a unique solution through every point except those on the x-axis

(where $\frac{-x}{y} \to \infty$)

→ But solutions are not defined for all x. Domain of definition depends on which solution we are looking at.

Separable equation: $\frac{dy}{dx} = f(x,y) = F(x)\,G(y)$

function of x only

function of y only

$$\frac{dy}{dx} = F(x)\,G(y)$$

$$\frac{dy}{G(y)} = F(x)\,dx \to \int \frac{dy}{G(y)} = \int F(x)\,dx \text{ etc.}$$

We could also write the equation as

$$M(x) + N(y)\frac{dy}{dx} = 0$$

$$\left[\begin{array}{ll} M(x) = -F(x) & \\ N(y) = 1/G(y) & \dfrac{dy}{dx} = \dfrac{-M(x)}{N(y)} \end{array}\right]$$

Let's try to integrate $M(x) + N(y)\frac{dy}{dx} = 0$

Let: $H_1(x)$ be antiderivative of $M(x)$: $H_1'(x) = M(x)$

$H_2(y)$ be antiderivative of $N(y)$: $H_2'(y) = N(y)$

Consider

$$\frac{d}{dx}\left[H_1(x) + H_2(y)\right] = \frac{d}{dx}\left[H_1(x)\right] + \frac{d}{dx}\left[H_2(y)\right]$$

$$= H_1'(x) + H_2'(y)\frac{dy}{dx} = M(x) + N(y)\frac{dy}{dx}$$

$= 0$ whenever $y(x)$ is a solution.

So the DE is equivalent to

$$\frac{d}{dx}\left[H_1(x) + H_2(y)\right] = 0$$

or $H_1(x) + H_2(y) = C$ constant

- $H_1(x) + H_2(y)$ is constant along any solution curve

[Kind of like a conservation law
Conservation of $H_1(x) + H_2(y)$]

Eg. $\frac{dy}{dx} = \frac{2x}{3y^2+1} \rightarrow (3y^2+1)\frac{dy}{dx} = 2x$

$$\underbrace{-2x}_{M(x)} + \underbrace{(3y^2+1)}_{N(y)}\frac{dy}{dx} = 0$$

$H_1(x) = -x^2$ $\qquad H_1(x) + H_2(y) = C$

$H_2(y) = y^3+y$ $\qquad -x^2 + y^3 + y = C$

$x^2 = y^3 + y - C$

Are solutions defined for all x?

→ implicit equation for y as a function of x.

→ Need to solve for y
There may be several solutions or no solutions.

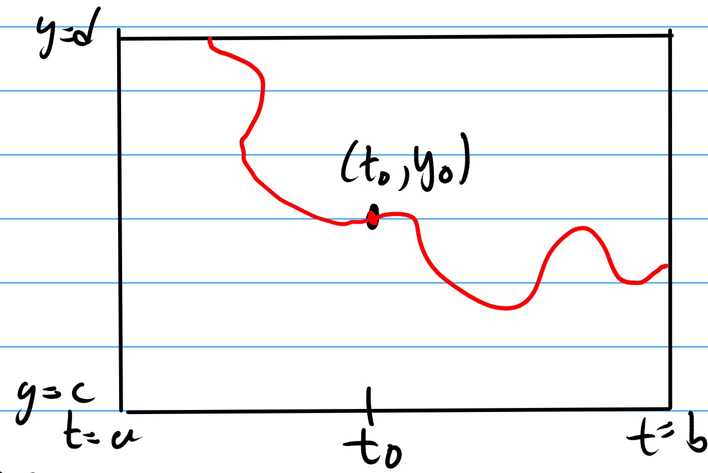
Ex $\frac{dy}{dx} = \frac{y}{x(1+y)} \Rightarrow \ln y + y - \ln x = \text{const}$

$cx = ye^y$

Theorem (Existence and Uniqueness for 1st order ODE)

Consider initial value problem

$$y' = f(t,y)$$
$$y(t_0) = y_0$$

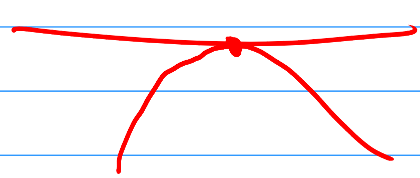


Suppose $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are continuous in the rectangle

$$a < t < b$$
$$c < y < d$$

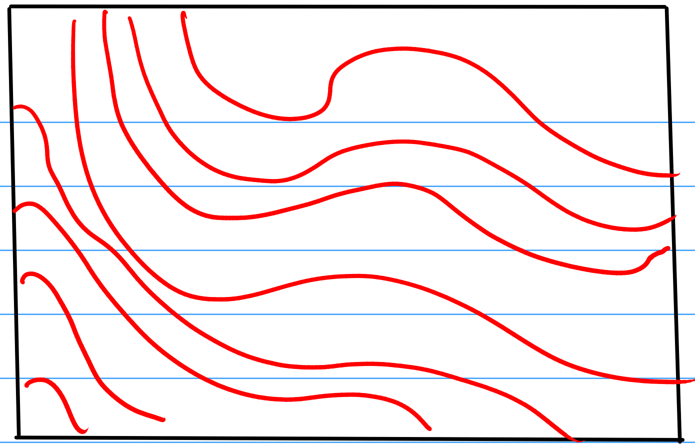
Then the initial value problem has a unique solution within this rectangle

- Explains why solution curves can't be tangent



violates uniqueness

- Solutions exist until they hit a discontinuity of f (but don't know when that will be a priori)



unique solutions thru every point

Discontinuities of $\frac{\partial f}{\partial y}$

Hourglass

Volume $= y^{2/3}$

$\frac{dV}{dt} = -1$ sand is draining

$\frac{2}{3} y^{-1/3} \frac{dy}{dt} = -1$

$\frac{dy}{dt} = -\frac{3}{2} y^{1/3}$ if $y > 0$

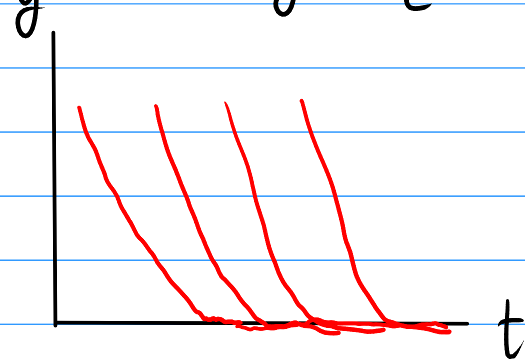
$\frac{dy}{dt} = 0$ if $y = 0$

$\frac{dy}{dt} = f(y)$

f is continuous

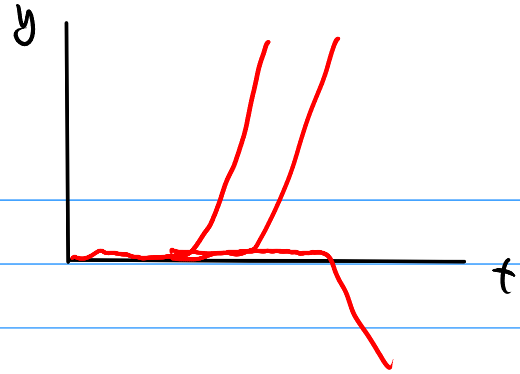
Solution $y^{2/3} = -t + c = c - t \quad t < c$

$y = [c-t]^{3/2} \quad t < c$



Does not satisfy backward uniqueness

$$\frac{dy}{dt} = \frac{3}{2} y^{1/3}$$

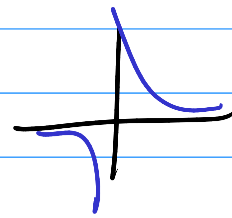


Does not satisfy forward uniqueness

Why doesn't this contradict the theorem?

$f(t,y) = \frac{3}{2} y^{1/3}$ continuous

$\frac{\partial f}{\partial y}(t,y) = \frac{1}{2} y^{-2/3}$



Discontinuous