Final Exam Friday December 14 9am–12noon
JGB 2.216

Practice Exam
Final is comprehensive
There will be some emphasis on Chapter 10.

<u>Exam rules</u>: Table of Laplace transforms will be provided
- Allowed one sheet (both sides) of notes
- Handwritten and no photocopies
- 8.5" x 11" sheet of paper.

Kenny's Office hours: Monday 12/10 2–4pm
Thursday 12/13 12–2pm
JP's office hours: Tuesday 12/11 2–4pm
Wednesday 12/12 2–4pm
Please fill out course evaluation.

Review Chapter 10

Boundary value problems in 1-dimension
$y''+y=0 \qquad y(0)=0 \qquad y(\pi)=0$

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}$$

Heat equation and Wave Equation.

Eg. Solve eigenvalue-eigenfunction problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0$$

$r^2 + \lambda = 0$

Definition of eigenvalue: a value of λ so that the problem has a nonzero solution $y(x)$

Consider $\lambda > 0$ general solution of $y'' + \lambda y = 0$

$r^2 + \lambda = 0$

$r^2 = -\lambda$ $\qquad y = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$

$r = \pm\sqrt{\lambda}\, i$

$y' = c_1\sqrt{\lambda}(-1)\sin\sqrt{\lambda}\, x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\, x$

$0 = y'(0) = -c_1\sqrt{\lambda}\sin 0 + c_2\sqrt{\lambda}\cos 0 = c_2\sqrt{\lambda}$

$\Rightarrow c_2 = 0$

just have $y = c_1\cos\sqrt{\lambda}\, x$

$0 = y'(\pi) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}\,\pi$

Eigenfunction: require $y(x)$ is not always $= 0$.

$\Rightarrow c_1 \neq 0$

$\Rightarrow \sin\sqrt{\lambda}\,\pi = 0$

$\sin n\pi = 0$ means need $\sqrt{\lambda} = n$ or $\lambda = n^2$

Eigenvalues $\lambda_n = n^2 \quad n = 1, 2, 3, \ldots$

Eigenfunctions $y_n = c_1\cos nx$

Consider $\lambda = 0$ $\quad y'' = 0 \quad y'(0) = 0 \quad y'(\pi) = 0$

$y = c_1 x + c_2 \quad y' = c_1 \quad y'(0) = 0 \Rightarrow c_1 = 0$

$y'(\pi) = 0 \Rightarrow c_1 = 0$

require y not always $= 0$, get $y = c_2$ for any non zero constant c_2 $\quad \lambda_0 = 0$ is an eigenvalue

eigenfunctions $y_0 = c$ constant.

Consider $\lambda < 0$ $\quad y'' + \lambda y = 0 \quad y'(0) = 0 = y'(\pi)$

$r^2 + \lambda = 0$

$r^2 = -\lambda \leftarrow$ is a positive number

$r = \pm\sqrt{-\lambda}$ real number.

$y = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$

$y' = c_1\sqrt{-\lambda}\, e^{-\sqrt{-\lambda}x} + c_2(-\sqrt{-\lambda})\, e^{-\sqrt{-\lambda}x}$

$0 = y'(0) = c_1\sqrt{-\lambda} - c_2\sqrt{-\lambda} = \sqrt{-\lambda}(c_1 - c_2)$

$\Rightarrow c_1 - c_2 = 0 \quad \Rightarrow c_1 = c_2$

$y = c_1\left(e^{-\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}\right)$

$y' = c_1\sqrt{-\lambda}\left(e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}\right)$

$0 = y'(\pi) = c_1\sqrt{-\lambda}\left(e^{\sqrt{-\lambda}\pi} - e^{-\sqrt{-\lambda}\pi}\right)$

$e^{\sqrt{-\lambda}\pi} = e^{-\sqrt{-\lambda}\pi}$

$\sqrt{-\lambda}\pi = -\sqrt{-\lambda}\pi$ impossible $\Rightarrow$ conclude $c_2 = 0$

Every $\lambda < 0$ is NOT an eigenvalue

Fourier series $f(x)$ periodic (period $2L$)

Want to write $f(x)$ as a trigonometric series:

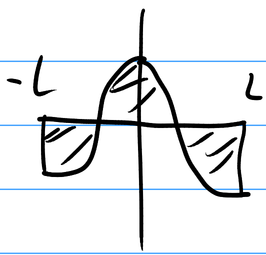
$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}$$

There is a formula for a_m, b_m but you don't always need to use it

$$L = 2 \qquad f(x) = \underbrace{15\cos\frac{3\pi x}{2}}_{a_3 \cos\frac{3\pi x}{2}} + \underbrace{9\sin 5\pi x}_{= 9\sin\frac{10\pi x}{2}} + \underbrace{1}_{=\frac{a_0}{2}}$$

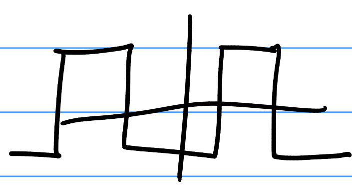
$$b_{10}\sin\frac{10\pi x}{L}$$

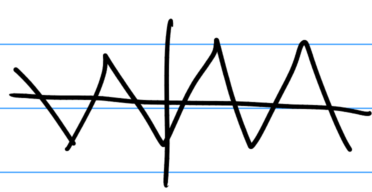
$$\boxed{a_3 = 15} \qquad \boxed{b_{10} = 9} \qquad \boxed{a_0 = 2}$$



General: $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \qquad b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx$$

Square-type wave  this one is doable

Triangle wave  requires integration by parts

Wave equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$

Example on $0 < x < L$, $0 < t$

Boundary conditions $u(0,t) = 0 \quad u(L,t) = 0$

Initial conditions $\frac{\partial u}{\partial t}(x,0) = 0 \quad u(x,0) = f(x)$

where $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi x}{L}$

Recall fundamental solutions

$$u_n(x,t) = \cos \frac{n\pi a t}{L} \sin \frac{n\pi x}{L}$$

General solution: $u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n \cos \frac{n\pi a t}{L} \sin \frac{n\pi x}{L}$

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}$$

In order to satisfy $u(x,0) = f(x)$, set $c_n = \frac{1}{n^2}$

solution: $u(x,t) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{n\pi a t}{L} \sin \frac{n\pi x}{L}$

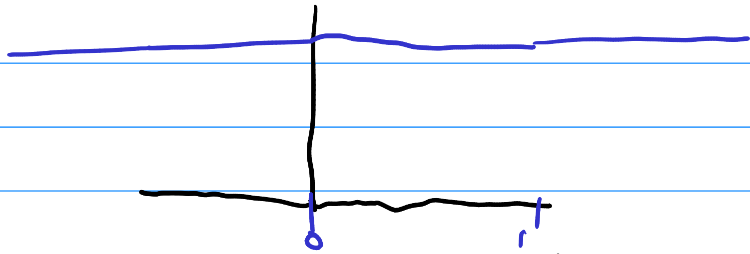
Cosine and sine series: $f(x)$ defined on $0 < x < L$

Represent as a trigonometric series.

1. extend $f(x)$ to an odd function on $-L < x < L$ then extend to $2L$-periodic function, take Fourier series. (Only sines)
2. extend $f(x)$ to an even function on $-L < x < L$ then extend to $2L$-periodic, take Fourier series (only cosines and possibly constant term)

$f(x) = 1$ for $0 < x < 1$

For cosine series consider this:

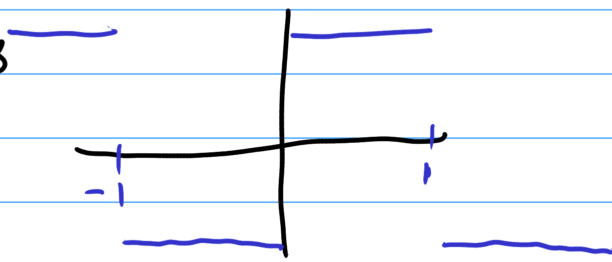


ie., we define $f(x) = 1$ everywhere.

$$f(x) = \frac{a_0}{2} + \sum \underbrace{\cancel{a_m}}_{0} \cos m\pi x + \underbrace{\cancel{b_m}}_{0} \sin m\pi x$$

$a_0 = 2$

Find sine series



odd $\Rightarrow$ no constant or cosine terms

$$f(x) = \sum b_m \sin m\pi x \qquad b_m = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{m\pi x}{L}\, dx$$

$L = 1$

$$b_m = \int_{-1}^{1} f(x) \sin m\pi x\, dx = 2\int_{0}^{1} f(x) \sin m\pi x\, dx$$

$$= 2\int_{0}^{1} \sin m\pi x\, dx = \frac{2}{m\pi}\left[1 - \cos m\pi\right]$$

$$= \frac{2}{m\pi}\begin{cases} 0 & \text{if } m \text{ even} \\ 2 & \text{if } m \text{ odd} \end{cases} = \begin{cases} 0 & \text{if } m \text{ even} \\ \frac{4}{m\pi} & \text{if } m \text{ odd} \end{cases}$$

$$f(x) = \frac{4}{\pi} \sum_{m \text{ odd}} \frac{1}{m} \sin m\pi x$$