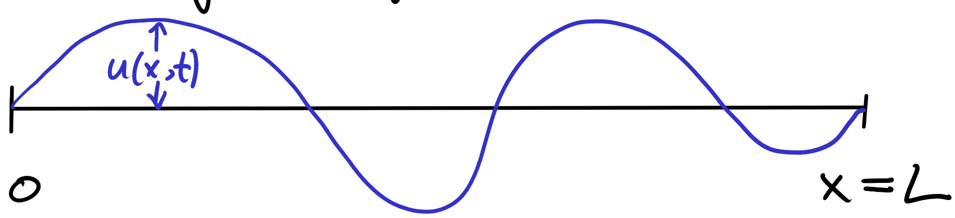
The Wave Equation: $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$

u is a function of position, x, and of time, t.

u could represent any number of physical quantities,
- → displacement of air (sound waves)
- → displacement of water (water waves)
- → Electromagnetic fields (EM waves i.e. radio, light, X-ray, etc.)

Main case for today:
u represents the vertical displacement of a vibrating string



We'll use separation of variables and Fourier series, similar to what we did for the heat equation

(Heat $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ vs wave $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$
very similar but wave is 2nd order in time)

Actually, before doing the vibrating string, I'd like to mention a very large class of solutions

"Free wave propogation"
Domain of x = whole line $-\infty < x < \infty$
→ No boundary conditions

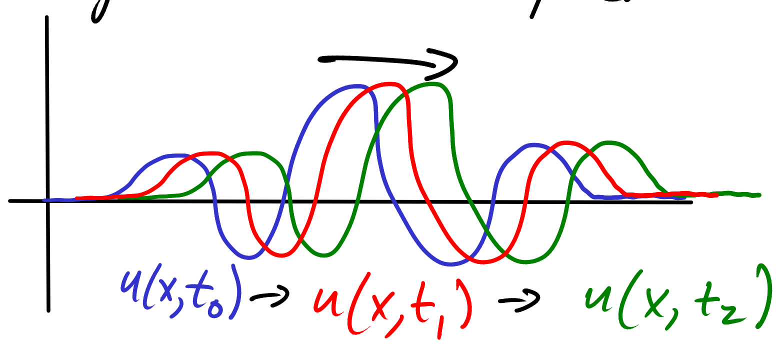
Let $f(s)$ be any twice differentiable function and consider

$$u(x,t) = f(x-at)$$

$$\frac{\partial^2 u}{\partial t^2} = (-a)^2 f''(x-at) = a^2 \frac{\partial^2 u}{\partial x^2}$$

So $u(x,t)$ solves wave equation: "Right-moving solution"

$f(s)$ is a "waveform", and plugging in $s = x-at$ makes a wave that keeps this shape and translates to the right with a speed a.



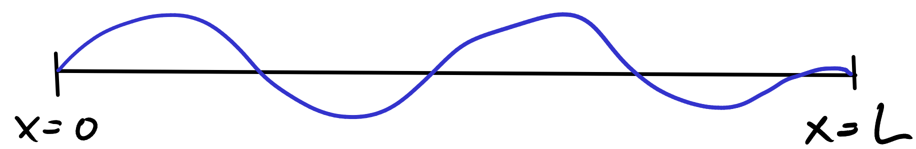
Similarly $f(x+at)$ is a left-moving solution

General free wave propogation

$$u(x,t) = \underset{\text{(right mover)}}{f(x-at)} + \underset{\text{(left mover)}}{g(x+at)}$$

This reveals that the parameter a is related to the speed of the waves (speed of sound, speed of light, etc)

Vibrating String (a = speed of wave propagation in string)



Domain of x: $0 < x < L$ = length of string

Boundary conditions: $u(0,t) = 0$, $u(L,t) = 0$

Ends of string are tied down, eg. string is stretched along a guitar

Initial conditions: $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$ } Need both because equation is second order in t.

To find fundamental solutions, use separation of variables

Write $u(x,t) = X(x)\, T(t)$

$$\frac{\partial^2 u}{\partial t^2} = X\, T'' \qquad \frac{\partial^2 u}{\partial x^2} = X''\, T$$

(prime denotes deriv w.r.t. x or t as appropriate)

Wave eqn: $$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$X\, T'' = a^2 X'' T$$

$$\underbrace{\frac{X''}{X}}_{\text{function of } x} = \underbrace{\frac{1}{a^2}\frac{T''}{T}}_{\text{function of } t}$$

As with Heat equation, we now argue:

Since LHS doesn't depend on t, RHS doesn't either
Since RHS doesn't depend on x, LHS doesn't either

So Both sides are equal to a constant, call it $-\lambda$

$$\frac{X''}{X} = -\lambda = \frac{1}{a^2}\frac{T''}{T}$$

Two equations $\begin{cases} X'' + \lambda X = 0 \\ T'' + a^2 \lambda T = 0 \end{cases}$

The boundary conditions $u(0,t) = 0 \quad u(L,t) = 0$
are satisfied by requiring $X(0) = 0 \quad X(L) = 0$.

So here $\begin{cases} X'' + \lambda X = 0 \ , \ X(0) = 0, X(L) = 0 \\ T'' + a^2 \lambda T = 0 \end{cases}$

The equation for X is the exact same boundary value - eigenvalue-eigenfunction problem we saw in §10.1 and last time in §10.5.

Solution: eigenvalues $\quad \lambda_n = (n\pi/L)^2 \quad (n = 1, 2, 3, \ldots)$
eigenfunctions $\quad X_n = \sin \frac{n\pi x}{L}$

Using these eigenvalues, can solve T-equation:

$$T'' + a^2 \lambda_n T = 0$$

$$T'' + a^2 \left(\frac{n\pi}{L}\right)^2 T = 0$$

$$\Rightarrow \quad T(t) = k_1 \cos \frac{n\pi a t}{L} + k_2 \sin \frac{n\pi a t}{L}$$

So $u(x,t) = \left(k_1 \cos \frac{n\pi a t}{L} + k_2 \sin \frac{n\pi a t}{L}\right) \sin \frac{n\pi x}{L}$

Solves $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial u}{\partial x^2}$, $u(0,t) = 0$, $u(L,t) = 0$.

Now we need to get the initial condition $\begin{cases} u(x,0) = f(x) \\ \frac{\partial u}{\partial t}(x,0) = g(x) \end{cases}$ into the story.

For simplicity, we only consider the case where

$$\frac{\partial u}{\partial t}(x,0) = 0$$

This means string is "plucked": stretched into shape given by $u(x,0) = f(x)$, then released and allowed to vibrate.

We can impose $\frac{\partial u}{\partial t}(x,0) = 0$ in our seperated solution by imposing $T'(0) = 0$

$$T = k_1 \cos \frac{n\pi a t}{L} + k_2 \sin \frac{n\pi a t}{L}$$

$$T' = k_1 \frac{n\pi a}{L}\left(-\sin\left(\frac{n\pi a t}{L}\right)\right) + k_2 \frac{n\pi a}{L} \cos \frac{n\pi a t}{L}$$

$0 = T''(0) = k_2 \frac{n\pi a}{L}$ so need $\boxed{k_2 = 0}$.

Thus our separable solution is $u(x,t) = k_1 \cos \frac{n\pi a t}{L} \sin \frac{n\pi x}{L}$

Set $k_1 = 1$ to get $u_n(x,t) = \cos \frac{n\pi a t}{L} \sin \frac{n\pi x}{L}$

This solution is called the nth <u>natural mode</u> of vibration

It has spatial period $\frac{2L}{n}$ = "wavelength"

It has temporal period $\frac{2L}{na}$ and temporal frequency $\frac{na}{2L}$

When a string vibrates, it is the temporal frequencies that we hear.

In order to satisfy initial displacement condition, can't use just one natural mode, but some combination of them:

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n \cos \frac{n\pi a t}{L} \sin \frac{n\pi x}{L}$$

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}$$

So to satisfy $u(x,0) = f(x)$, let c_n be the coefficients of the Fourier sine series for $f(x)$

$$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \qquad \left(\text{Fourier sine series coefficients} \right)$$

Recall: The Fourier sine series is defined as follows:

→ Extend $f(x)$ on $0<x<L$ to an odd function on $-L<x<L$, then extend to $2L$-periodic function, and take the Fourier series for this extention. Since the function is odd, the fourier series has only sines.

→ Thus it gives a representation of $f(x)$ on $0<x<L$ using only sines.

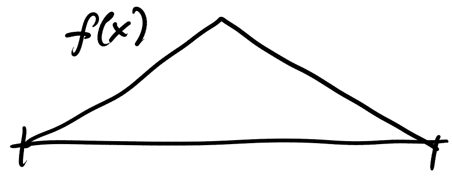
Thus if $f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}$

then $u(x,t) = \sum_{n=1}^{\infty} c_n \cos \frac{n\pi a t}{L} \sin \frac{n\pi x}{L}$

- Decompose $f(x)$ into spatial sine terms to determine c_n = strength of nth natural mode.

- Each natural mode vibrates with temporal frequency $\frac{na}{2L}$.

Eg. solve when $L=20$ and $f(x) = \begin{cases} x & 0<x<10 \\ 20-x & 10<x<20 \end{cases}$



"plucked string"

$$c_n = \frac{2}{20}\int_0^{20} f(x) \sin \frac{n\pi x}{20}\, dx = \frac{2}{20}\left(\frac{3200}{n^2\pi^2}\right)\left(\frac{1}{4}\right)\begin{cases} (-1)^{k+1} & n=2k-1 \\ 0 & n=2k \end{cases}$$

(Mathematica)

$$= \frac{80}{n^2\pi^2}\begin{Bmatrix} (-1)^{k+1} & n=2k-1 \\ 0 & n=2k \end{Bmatrix}$$

So $f(x) = \frac{80}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^2} \sin \frac{(2k-1)\pi x}{20}$

$$u(x,t) = \frac{80}{\pi^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)^2} \cos \frac{(2k-1)\pi a t}{20} \sin \frac{(2k-1)\pi x}{20}$$

If we knew eg. $a = 10$ we could plug that in as well.