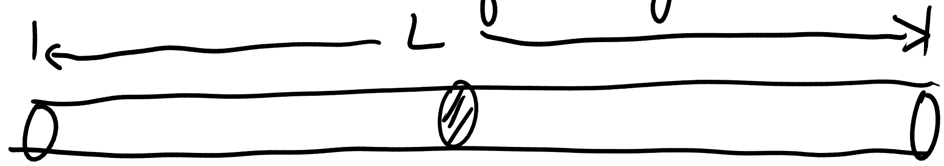
Heat Conduction in one dimension

Consider a thin rod of length L



cross section area is very small.
→ rod is essentially 1-dimension

made of a material with varying temperature.

Temperature of rod tends toward environmental temp
as in Newton's law of cooling: $\frac{dT}{dt} = k(T_e - T)$

This law is large-scale / coarse.

This law assumes all parts of the rod have same temp. at any given time. ← Not true in today's lecture.

Today: assume the sides of rod are insulated
Thermal energy / Heat can only flow in/out through ends of rod.

Temperature depends on position as well as time.

Describe temp. by a function $u(x,t)$

$0 \leq x \leq L$

$0 \leq t < \infty$

Physics $\rightsquigarrow$ $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}$

α^2 is a positive constant that depends on the material.

Thermal Diffusivity

silver	$\alpha^2 = 1.71$	cm^2/s
copper	$\alpha^2 = 1.14$	
aluminum	$\alpha^2 = 0.86$	

larger $\alpha^2 \Longleftrightarrow$ faster heat diffusion

Boundary / Initial conditions

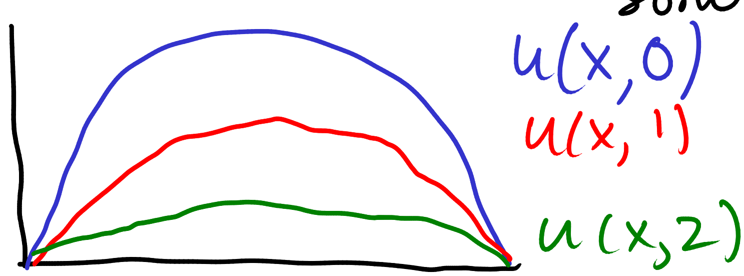
End of rod held at $0°C$

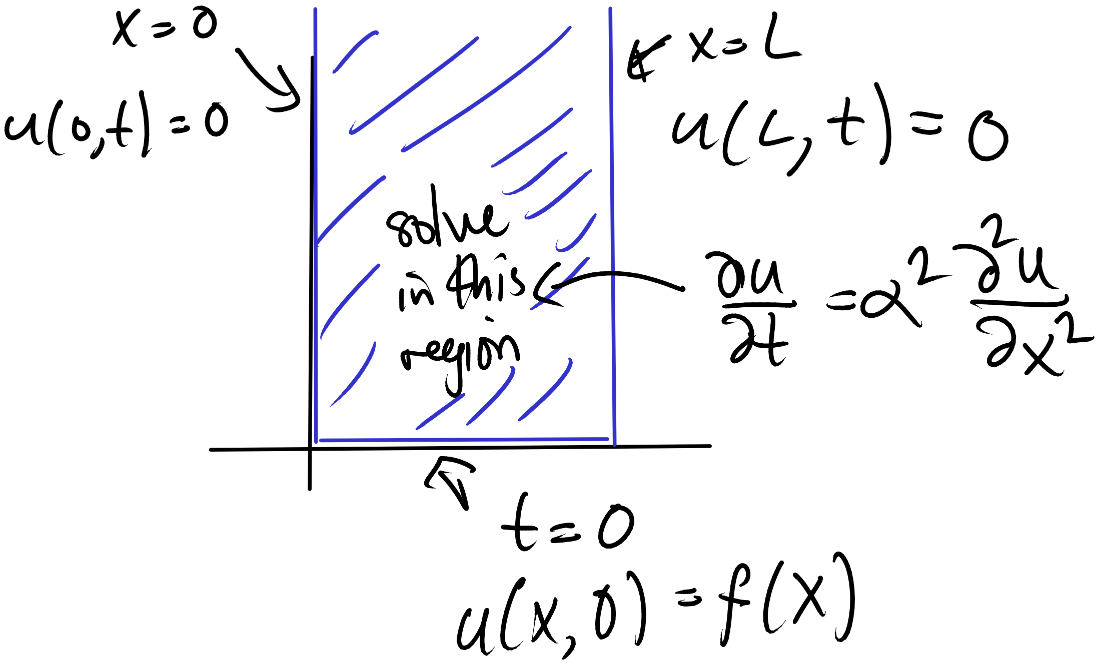
Left end: $u(0,t) = 0$ for all t

Right end: $u(L,t) = 0$ for all t.

Initial temp distribution $u(x,0) = f(x)$ where $f(x)$ is some given function.

looks like





Separation of variables: Try to break the partial differential equation into several ODE's.

Assume $u(x,t) = X(x)\,T(t)$ is a product of functions one $X(x)$ depends only on x, $T(t)$ depends only on t.

(the assumption will be justified if we find interesting solutions of this form).

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(X(x)T(t)\right) = X(x)\frac{\partial T}{\partial t}(t)$$

$$= X T'$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(X(x)\,T(t)\right) = \frac{\partial^2 X}{\partial x^2}(x)\,T(t)$$

$$= X'' T$$

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \iff X T' = \alpha^2 X'' T$$

$$\Longleftrightarrow \quad \underbrace{\frac{X''}{X}}_{\substack{\text{function} \\ \text{of } x \text{ only}}} = \underbrace{\frac{1}{\alpha^2}\frac{T'}{T}}_{\substack{\text{function} \\ \text{of } t \text{ only}}} \Rightarrow \text{Both sides are constant}$$

Let's call this constant $-\lambda$

$$\Longleftrightarrow \quad \frac{X''}{X} = -\lambda = \frac{1}{\alpha^2}\frac{T'}{T}$$

$\to$ λ can be any number but for any particular solution it will be constant.

$$X'' + \lambda X = 0$$

$$T' + \alpha^2 \lambda T = 0$$

$$0 = u(0,t) = X(0)\,T(t) \Longrightarrow X(0) = 0$$

$$0 = u(L,t) = X(L)\,T(t) \Longrightarrow X(L) = 0$$

Let's focus on X:

$$X'' + \lambda X = 0 \qquad \begin{array}{l} X(0) = 0 \\ X(L) = 0 \end{array}$$

See section 10.1: eigenvalue-eigenvector problem with boundary conditions in one variable

Eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n = 1, 2, 3, \ldots$

Eigenfunctions $X_n = \sin \frac{n\pi x}{L}$

let's find $T(t)$ $T' + \alpha^2 \lambda T = 0$

$$T(t) = c e^{-\alpha^2 \lambda t}$$

For each λ_n, have a solution

$\lambda_n = \left(\frac{n\pi}{L}\right)^2 = \frac{n^2\pi^2}{L^2}$ $X_n = \sin \frac{n\pi x}{L}$ $T_n = c e^{-\alpha^2 \frac{n^2\pi^2}{L^2} t}$

$$u_n(x,t) = X_n T_n = e^{-\alpha^2 n^2 \pi^2 t / L^2} \sin \frac{n\pi x}{L}$$

Check it solves $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

$u_n(x,t)$ form a fundamental set of solutions to heat equation

For any constants c_n,

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2 n^2 \pi^2 t / L^2} \sin \frac{n\pi x}{L}$$

is also a solution of the heat equation, satisfying boundary conditions $u(0,t) = 0$
$u(L,t) = 0$

How to make $u(x,t)$ satisfy $u(x,0) = f(x)$ for a given function $f(x)$?

Note $u(x,t)$ is generally not a product XT

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2 n^2 \pi^2 t / L^2} \sin \frac{n\pi x}{L}$$

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}$$ this is a fourier sine series.

supposed to $= f(x)$

so $\sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} = f(x)$ is the sine series for $f(x)$.

$f(x)$ is a function on $0 < x < L$
extend to an odd function on $-L < x < L$
and take fourier series.

$$c_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

Example:

initial temp $= 100°C$ (constant) ends are held at $0°C$
$\alpha = 1$

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $u(0,t) = 0$ $u(x,0) = 100$
$u(50,t) = 0$ $0 < x < L$

General sol$^{\underline{n}}$: $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t/2500} \sin \frac{n\pi x}{50}$

$f(x) = 100$

$$C_n = \frac{2}{50}\int_0^{50} 100 \sin\frac{n\pi x}{50}\,dx = \frac{400}{n\pi} \text{ if } n \text{ is odd}$$

$$= 0 \text{ if } n \text{ is even}$$

constant function 100 $0 < x < 50$

$$= \frac{400}{\pi}\sum_{n=1,3,5,\ldots} \frac{1}{n}\sin\frac{n\pi x}{50}$$

$$u(x,t) = \frac{400}{\pi}\sum_{\substack{n=1,3,5,\ldots\\(n \text{ odd})}} \frac{1}{n} e^{-n^2\pi^2 t/2500}\sin\frac{n\pi x}{50}$$