Fourier Series Cont'd: Convergence, Even and Odd functions

Remember to Fill Out Course Evaluation!

Quiz tomorrow!

Recall: Fourier series $f(x)$ periodic

Period $2L$: $f(x+2L) = f(x)$

Represent $f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L}$

Orthogonality relations:

$$\int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} \, dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$$

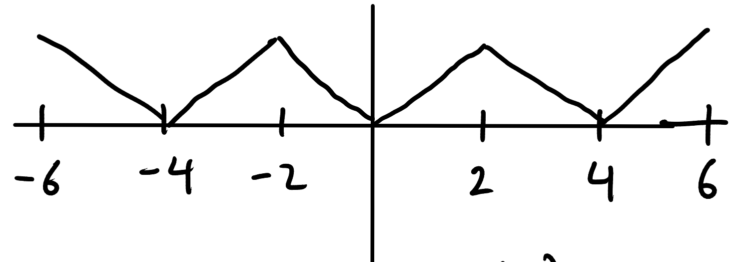
$$\int_{-L}^{L} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} \, dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$$

$$\int_{-L}^{L} \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L} \, dx = 0$$

$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,dx$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\,dx$

Example $f(x)$:



On the interval $-2 < x < 2$ $f(x)$ is given by

$$f(x) = \begin{cases} -x & -2 < x < 0 \\ x & 0 \le x < 2 \end{cases}$$

$\boxed{L=2}$

$f(x+4) = f(x)$
4-periodic

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$$

$$= \frac{1}{2}\int_{-2}^{0}(-x)\,dx + \frac{1}{2}\int_{0}^{2} x\,dx = 2$$

$$a_m = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{m\pi x}{2}\,dx$$

$$= \frac{1}{2}\left\{\int_{-2}^{0}(-x)\cos\frac{m\pi x}{2}\,dx + \int_{0}^{2} x\cos\frac{m\pi x}{2}\,dx\right\}$$

$$G(x) = \int x\cos\frac{m\pi x}{2}\,dx = \left(\frac{2}{m\pi}\right) x\sin\frac{m\pi x}{2} + \left(\frac{2}{m\pi}\right)^2\cos\frac{m\pi x}{2}$$

$$a_m = \frac{1}{2}\left\{- G(x)\Big|_{-2}^{0} + G(x)\Big|_{0}^{2}\right\}$$

$$= \frac{1}{2}\left\{-\left(\left(\frac{2}{m\pi}\right)^2 - \left(\frac{2}{m\pi}\right)^2\cos m\pi\right) + \left(\left(\frac{2}{m\pi}\right)^2\cos m\pi - \left(\frac{2}{m\pi}\right)^2\right)\right\}$$

$$= \left(\frac{2}{m\pi}\right)^2 \cos m\pi - \left(\frac{2}{m\pi}\right)^2 = \left(\frac{2}{m\pi}\right)^2 (\cos m\pi - 1)$$

$$\cos m\pi = (-1)^m \qquad = \left(\frac{2}{m\pi}\right)^2 ((-1)^m - 1)$$

m even $\quad a_m = 0$

m odd get $a_m = \left(\frac{2}{m\pi}\right)^2 (-2) = \frac{-8}{m^2\pi^2}$

Last thing: $b_m = \frac{1}{2}\int_{-2}^{2} f(x) \sin \frac{m\pi x}{L}\, dx$

this is actually $\underline{0}$.

The Fourier Series is

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos \frac{m\pi x}{2} = 1 - \frac{8}{\pi^2} \sum_{m \text{ odd}} \frac{\cos(m\pi x/2)}{m^2}$$

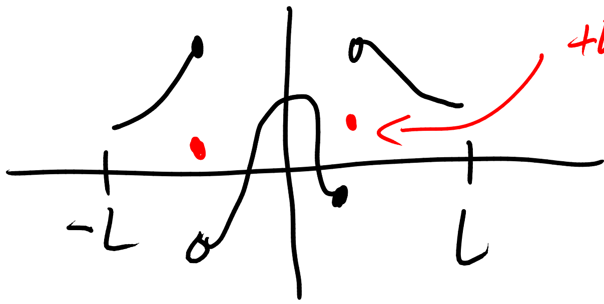
Every odd m is $2n-1$ for $n = 1, 2, 3$

$$1 - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos((2n-1)\pi x/2)}{(2n-1)^2}$$

Q: Does Fourier series converge to the function?

A: If f is $2L$-periodic and

suppose f and f' are piecewise continuous and have finite limits



the Fourier series converges to the average of the two limits

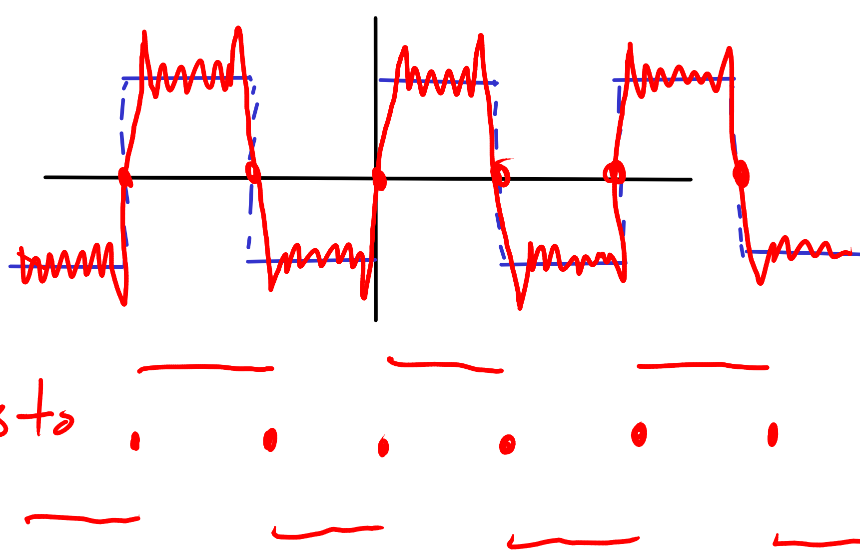
Then the Fourier series $\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}$

① converges to $f(x)$ at each point where $f(x)$ is continuous

② at each point of discontinuity x, it converges to

$$\frac{1}{2}\left[\lim_{z\to x^+} f(z) + \lim_{z\to x^-} f(z)\right]$$

Eg.



Red: partial sum

converges to

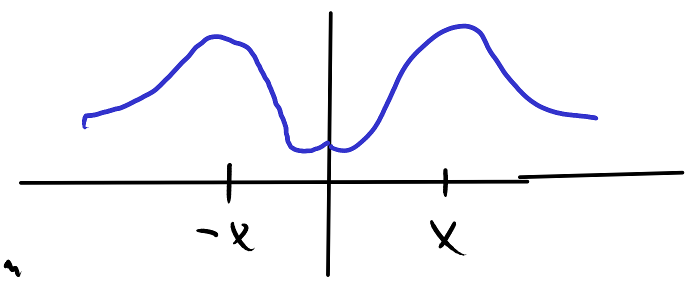
Even and odd function

<u>Even</u> : Symmetrical about y-axis

$$f(-x) = f(x)$$

Eg. x^2, x^4, x^6

$\cos \frac{m\pi x}{L}$

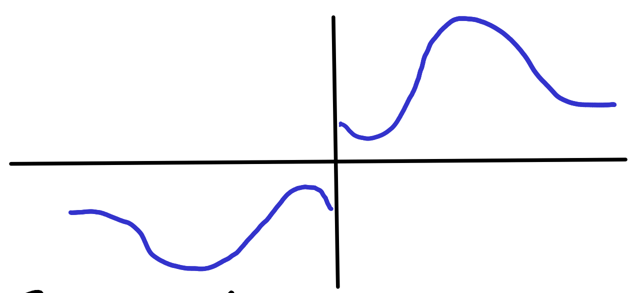


<u>Odd</u> : "Symmetrical about 180° rotation"

$$f(-x) = -f(x)$$

Eg x, x^3, x^5

$\sin \frac{m\pi x}{L}$



For functions

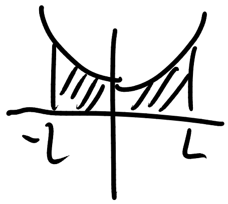
<u>Facts</u> :

even + even = even
odd + odd = odd } different from number
even x even = even
odd x odd = even
odd x even = odd

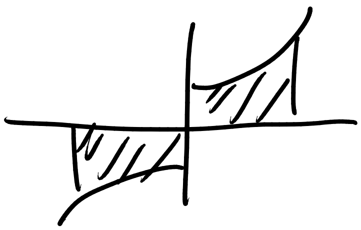
proof $\underset{\text{odd}}{f(-x)}\,\underset{\text{odd}}{g(-x)} = (-f(x))(-g(x)) = f(x)\,g(x)$

hence $f \times g$ is even.

odd + even = anything. (neither odd nor even) if f is even $\int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx$



if f is odd: $\int_{-L}^{L} f(x)\,dx = 0$



Suppose f is periodic, period $2L$, and even

Then $f(x)\cos\frac{m\pi x}{L}$ is even

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{m\pi x}{L}\,dx$$

Then $f(x)\sin\frac{m\pi x}{L}$ is odd

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx = 0$$

On the other hand if f is odd

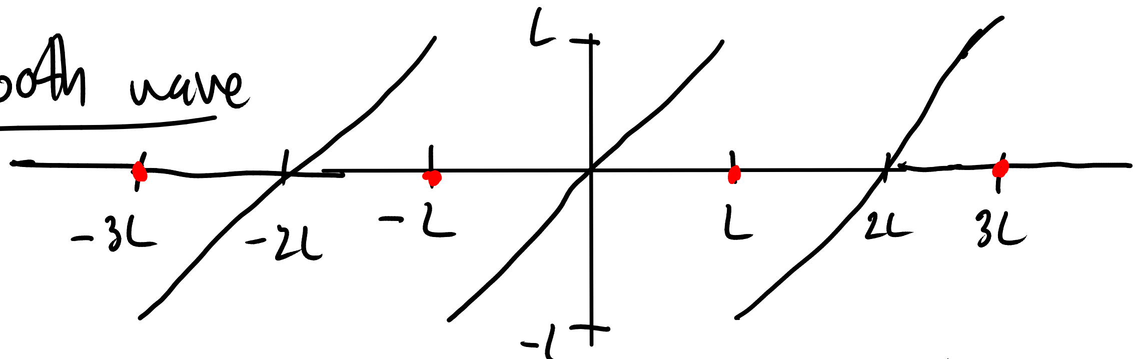
$f(x)\cos\frac{m\pi x}{L}$ is odd $a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = 0$

$f(x)\sin\frac{m\pi x}{L}$ is even $b_m = \frac{2}{L}\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx$

f is even $\Rightarrow$ Fourier series has only cosines (and constant term) "cosine series"

f is odd $\Rightarrow$ Fourier series has only sines (and no constant term) "sine series"

Sawtooth wave



odd: $a_n = 0$ just find the sine terms

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L x\sin\frac{n\pi x}{L}\,dx$$

$\rightsquigarrow$ integrate by parts

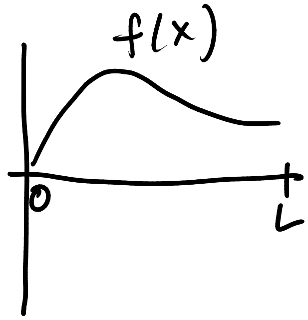
$$= \frac{2}{L}\left\{\left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi x}{L} - \left(\frac{L}{n\pi}\right)x\cos\frac{n\pi x}{L}\right\}\Bigg|_0^L$$

$$= \frac{2}{L}\left\{-\frac{L^2}{n\pi}\cos n\pi\right\} = \frac{2L}{n\pi}\left\{-\cos n\pi\right\}$$

$$= \frac{2L}{n\pi}\left\{-(-1)^n\right\} = \frac{2L}{n\pi}(-1)^{n+1}$$

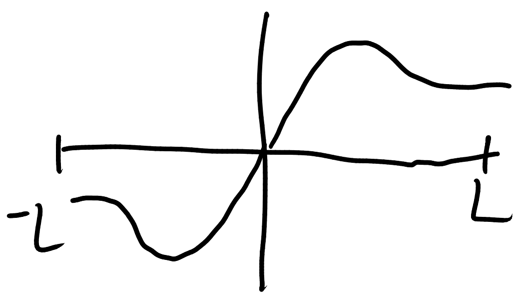
Fourier series $\displaystyle \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{L}$

Another thing Fourier series can do:
Represent a function on a fixed interval say $0 < x < L$



can be represented by a sine series or a cosine series:

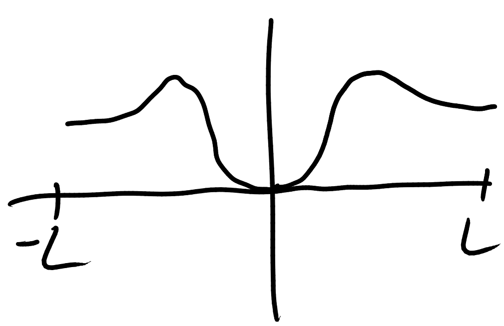
Sine series: extend $f(x)$ to be <u>odd</u> on $-L < x < L$



Take the sine series for this odd function

$$\sum_{m=1}^{\infty} b_m \sin \frac{m\pi x}{L}$$

OR Cosine series: extend $f(x)$ to be <u>even</u> on $-L < x < L$



Take the cosine series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos \frac{m\pi x}{L}$$