Exam 2 Stats mean 80.1

3rd Q 90

Median: 81

1st Q 73.25

Fourier Series:

Idea: write a function $f(x)$ as a sum of sines and cosine

Compare power series = write $f(x)$ as a sum of powers of $x, x^2, x^3, \dots$

Why? $y = \sin(\omega x)$ or $y = \cos(\omega x)$

$$y'' = -\omega^2 y \quad \text{i.e.} \quad y'' + \omega^2 y = 0$$

ω = angular frequency (radians/second)

$f = \frac{\omega}{2\pi}$ frequency. (cycles/second)

$T = \frac{1}{f} = \frac{2\pi}{\omega}$ Period (seconds/cycle)

The functions $\sin(\omega x)$ and $\cos(\omega x)$ are eigenfunctions of the differential operator

$$L = -\frac{d^2}{dx^2} \quad \text{with eigenvalue } \omega^2$$

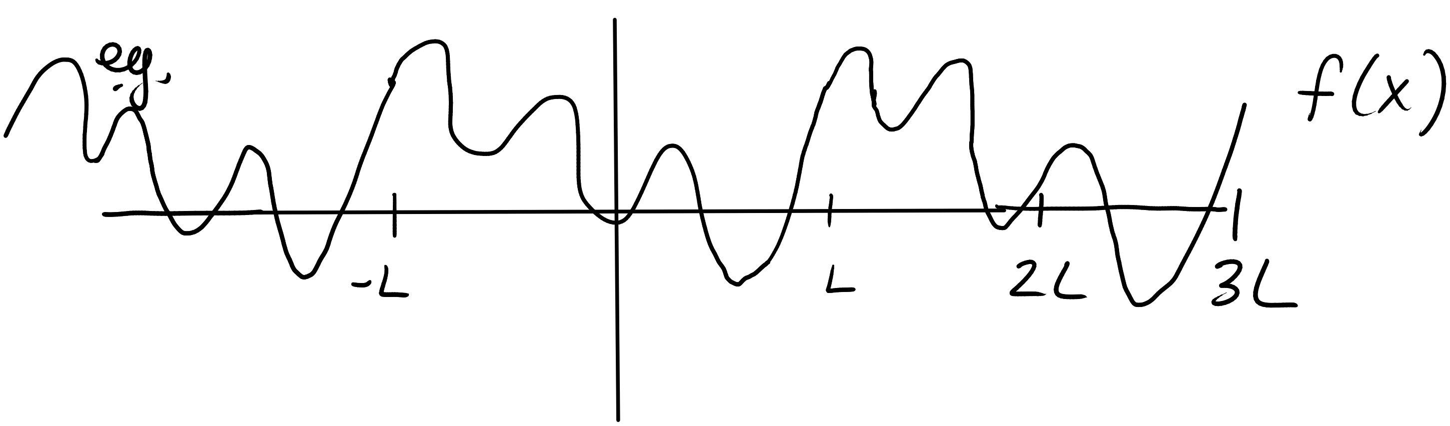
$$Ly = \omega^2 y \quad \text{that is,} \quad -\frac{d^2}{dx^2}y = \omega^2 y$$

Therefore they are useful, when studying

Heat equation: $\dfrac{\partial y}{\partial t} = \alpha^2 \dfrac{\partial^2 y}{\partial x^2}$

Wave equation: $\dfrac{\partial^2 y}{\partial t^2} = \alpha^2 \dfrac{\partial^2 y}{\partial x^2}$

Fourier series: take any form of oscillation:



$f(x)$ is a periodic function with period $2L$

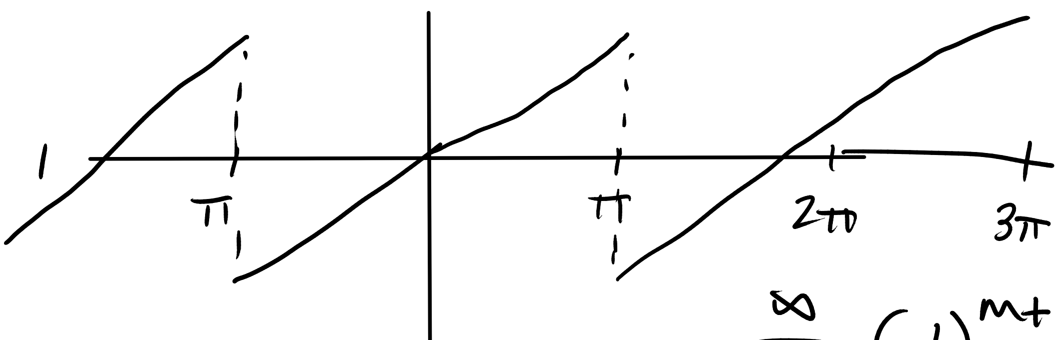
Want to decompose $f(x)$ into its <u>frequency components</u> (pure sinusoidal oscillation)

Def f is periodic if there is a fixed T

$$f(x+T) = f(x) \quad \text{for all } x$$

The smallest $T > 0$ that works in this formula is called the fundamental period

e.g. Take any function and let it repeat



Sawtooth wave

$$2 \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \sin(mx)$$

In general, we try to write (f periodic w/ period $2L$)

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right)$$

a_0, a_m, b_m are called Fourier <u>coefficients</u>

Fundamental Period of f $= 2L$

Fundamental frequency of $f = \frac{1}{2L}$

Frequency of $\sin \frac{m\pi x}{L}$

$= \frac{\left(\frac{m\pi}{L}\right)}{2\pi} = \frac{m}{2L}$

Frequencies in Fourier series are all integer multiples of the fundamental frequency of f.

<u>Question</u>: How to find Fourier Coefficients?

Idea: orthogonality relations

inner product / dot product for functions on an interval $\alpha \le x \le \beta$

$u(x)$ and $v(x)$ two functions on the interval $\alpha \le x \le \beta$

$$(u,v) = \int_{\alpha}^{\beta} u(x)\, v(x)\, dx$$

u and v are othogonal if $(u,v) = 0$

i.e. $$\int_{\alpha}^{\beta} u(x)\, v(x)\, dx = 0$$

Fact $\sin\frac{m\pi x}{L}$ and $\cos\frac{m\pi x}{L}$ on $-L \le x \le L$

are all mutually orthogonal.

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$$

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = 0$$

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \end{cases}$$

(period is $2L$)

Suppose $f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m\cos\frac{m\pi x}{L} + b_m\sin\frac{m\pi x}{L}$

To get a_n, take inner product with $\cos\frac{n\pi x}{L}$

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \int_{-L}^{L}\frac{a_0}{2}\cos\frac{n\pi x}{L}\,dx$$

$$+ \sum_{m=1}^{\infty} a_m\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx$$

$$+ b_m\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx$$

$$= a_n \cdot L \quad \text{by orthogonality}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$\int_{-L}^{L} f(x)\,dx = \frac{a_0}{2}\int_{-L}^{L} dx + \sum_{m=1}^{\infty} a_m\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx + b_m\int_{-L}^{L}\sin\frac{m\pi x}{L}\,dx$$

$$= \frac{a_0}{2}\int_{-L}^{L} dx = L a_0$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = L b_n$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

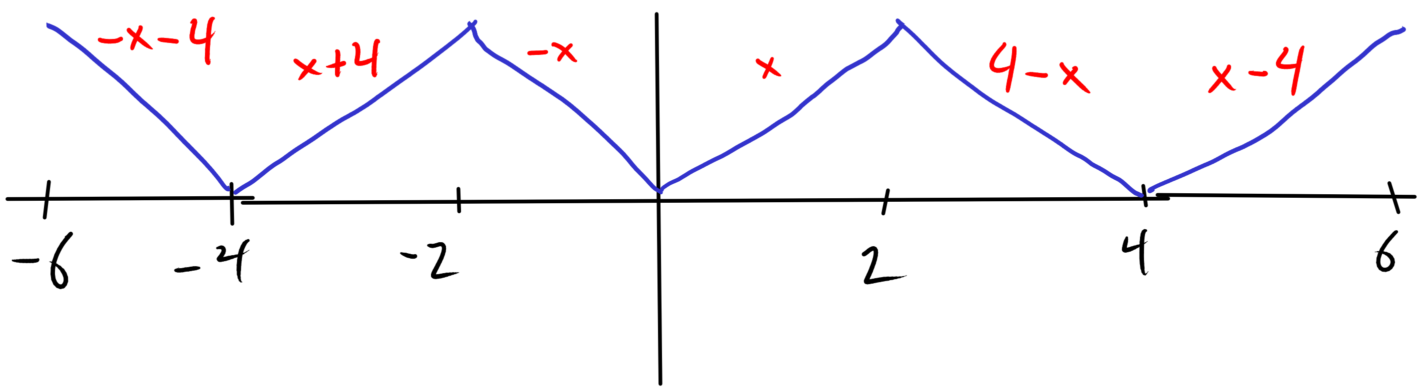
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

"Euler-Fourier formula for the coefficients"

<u>Ex</u> Find Fourier coefficients for $f(x)$

$$f(x) = \begin{cases} -x & -2 \le x < 0 \\ x & 0 \le x < 2 \end{cases}$$

and f is periodic with period 4.



Fundamental interval $-2 \le x \le 2$

$$2L = 2-(-2) = 4$$

$$L = 2$$

<u>constant coefficient</u>

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0}(-x)\,dx + \int_{0}^{2} x\,dx\right]$$

$$= 2$$

<u>mth cosine coefficient</u>

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{m\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0}(-x)\cos\frac{m\pi x}{2}\,dx + \int_{0}^{2} x\cos\frac{m\pi x}{2}\,dx\right]$$

$$\int x\cos(ax)\,dx = x\frac{1}{a}\sin(ax) - \int \frac{1}{a}\sin(ax)\,dx$$

$u = x \quad du = dx$

$dv = \cos(ax)dx$

$$= \frac{1}{a}x\sin(ax) - \frac{1}{a^2}(-\cos(ax))$$

$v = \frac{1}{a}\sin(ax)$

$$= \frac{1}{a}x\sin(ax) + \frac{1}{a^2}\cos(ax)$$

Apply with $a = \frac{m\pi}{2}$ . . . To be continued