Boundary Value Problems & Fourier Series

No Quiz Wednesday before Thanksgiving

Goal: understand Heat equation: $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

Wave equation: $\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

Boundary value problem for ODE:

Up to now, we've mainly talked about the initial value problem:

$$y'' + p(x)y' + q(x)y = g(x)$$

specify $y(x_0)$ and $y'(x_0)$

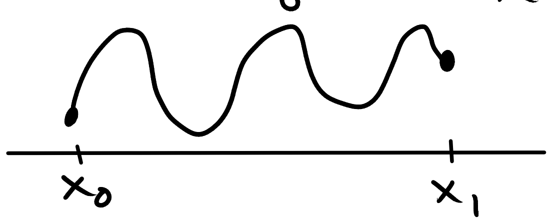
$\rightarrow$ this determines the unknown constants uniquely

Boundary value problem:

Solve $y'' + p(x)y' + q(x)y = g(x)$

specify $y(x_0)$ and $y(x_1)$

where x_0 and x_1 are different points.



That is, we specify the initial value, and the target final value, and we look for something that solves the differential equation in between.

E.g. $y'' + 2y = 0 \qquad y(0) = 0, \quad y(\pi) = 0$

General solution of the ODE.

$r^2 + 2 = 0 \qquad r = \pm\sqrt{2}\, i$

$y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x$

$0 = y(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 \Rightarrow c_1 = 0$

$0 = y(\pi) = \cancelto{0}{c_1} \cos\sqrt{2}\pi + c_2 \sin\sqrt{2}\pi = c_2 \sin\sqrt{2}\pi$

since $\sin\sqrt{2}\pi \neq 0$, we need $c_2 = 0$ as well

So the only solution is $y = 0$

E.g. $y'' + y = 0 \qquad y(0) = 0 \qquad y(\pi) = 0$

$y = c_1 \cos x + c_2 \sin x$

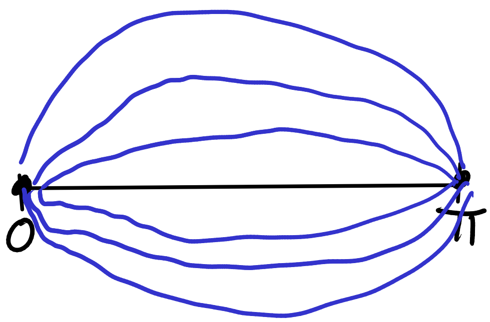
$0 = y(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 \Rightarrow c_1 = 0$

$0 = y(\pi) = \cancel{c_1 \cos}\, \pi + c_2 \sin\pi = c_2 \sin\pi$

since $\sin \pi = 0$, c_2 can be anything

there are infinitely many solutions $c \sin x$

all multiples of $\sin x$



Eg. $y'' + 2y = 0 \quad y(0) = 1, \quad y(\pi) = 0$

$$y = c_1 \cos \sqrt{2} x + c_2 \sin \sqrt{2} x$$

$$1 = y(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 \Rightarrow c_1 = 1$$

$$0 = c_1 \cos \sqrt{2}\pi + c_2 \sin \sqrt{2}\pi = \cos \sqrt{2}\pi + c_2 \sin \sqrt{2}\pi$$

$$c_2 \sin \sqrt{2}\pi = -\cos \sqrt{2}\pi$$

$$c_2 = -\cot \sqrt{2}\pi$$

Solution is : $y = \cos \sqrt{2} x - \cot \sqrt{2}\pi \sin \sqrt{2} x$

The solution exists and is unique.

Eg. $y'' + y = 0$, $y(0) = 1$, $y(\pi) = a$

$$y = c_1 \cos x + c_2 \sin x$$

$$1 = c_1 \cos 0 + c_2 \sin 0 = c_1 \Rightarrow c_1 = 1$$

$$a = 1 \cdot \underset{\cos\pi = -1}{\underbrace{\cos \pi}} + c_2 \underset{0}{\underbrace{\sin \pi}}$$

$a = -1$ and c_2 is anything.

If $a \neq -1$ in the statement of the problem, there is <u>NO SOLUTION</u>

If $a = -1$, there are infinitely many solutions.

$$y = \cos x + c_2 \sin x$$

$\cos 0 = 1$	$\sin 0 = 0$
$\cos \pi = -1$	$\sin \pi = 0$
satisfies boundary conditions.	adding a multiple of $\sin x$ doesn't spoil the boundary conditions.

$y'' + 2y = 0$ Boundary conditions at $0, \pi$

$\rightarrow$ always get unique solution.

$y'' + y = 0$ Boundary conditions at $0, \pi$

$\rightarrow$ sometimes no solution, sometimes infinitely many solutions.

Eigenvalues and Eigenfunctions.

the examples we've done are special cases of

$y'' + \lambda y = 0$ specify $y(0)$ and $y(\pi)$

when does this equation have interesting solutions?

i.e. find λ so that the equation has ∞-ly many solutions

(called an <u>eigenvalue</u>)

for each eigenvalue, describe the solutions:

(called <u>eigenfunctions</u>)

Analogy $A\vec{x} = \lambda \vec{x}$ for a matrix A.

Look for λ so that the system has a nonzero solution
(eigenvalues)

For each eigenvalue, find the nonzero solution (eigenvectors)

Consider the differential operator $D^2 = \frac{d^2}{dx^2}$

Just as you can multiply a matrix and a vector

you can "multiply" a differential operator and a function

$$D^2 y = \left(\frac{d^2}{dx^2}\right) y = y''$$

$$\left.\begin{aligned} D^2 y &= -\lambda y \\ D^2 y + \lambda y &= 0 \\ y'' + \lambda y &= 0 \end{aligned}\right\} \text{ eigenvalue problem for the operator } D^2 = \frac{d^2}{dx^2}$$

Solve eigenvalue problem $y'' + \lambda y = 0 \quad y(0) = 0 = y(\pi)$

$$r^2 + \lambda = 0$$

Cases $\lambda > 0$, $\lambda = 0$, $\lambda < 0$

Case $\lambda > 0$ $\quad \lambda = \mu^2 \quad r^2 + \mu^2 = 0$

$$r = \pm \mu i$$

$$y = c_1 \cos \mu x + c_2 \sin \mu x$$

$0 = y(0) = c_1 \cos 0 + c_2 \sin 0 = c_1 \Rightarrow c_1 = 0$

$0 = y(\pi) = \cancel{c_1 \cos \mu\pi} + c_2 \sin \mu\pi$

if $\sin \mu\pi \neq 0$, get $c_2 = 0$ as well

only solution is $y = 0$

if μ is not an integer, $\sin \mu\pi \neq 0$

and $\lambda = \mu^2$ is NOT an eigenvalue

if μ is an integer $\sin \mu\pi = 0$

and $y = c_2 \sin \mu x$ is a solution for any c_2

$\lambda = \mu^2$ is an eigenvalue, and

$y = c \sin \mu x$ is an eigenfunction for $\lambda = \mu^2$

<u>Case $\lambda = 0$:</u> $y'' + 0y = 0$ $\quad y(0) = 0 = y(\pi)$

$y = c_1 x + c_2$

$0 = y(0) = c_1 0 + c_2 = c_2 \Rightarrow c_2 = 0$

$0 = y(\pi) = c_1 \pi + c_2 = c_1 \pi \Rightarrow c_1 = 0$

$y = 0$ only solution. $\lambda = 0$ is NOT an eigenvalue.

Case $\lambda < 0$: $\quad y'' + \lambda y = 0 \quad y(0) = 0 = y(\pi)$

$\lambda = -\alpha^2 \qquad y'' - \alpha^2 y = 0$

$r^2 - \alpha^2 = 0 \qquad r = \pm\alpha$

$y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

$0 = y(0) = c_1 1 + c_2 1 \qquad c_1 = -c_2$

$0 = y(\pi) = c_1 e^{\alpha\pi} + c_2 e^{-\alpha\pi}$

$0 = -c_2 e^{\alpha\pi} + c_2 e^{-\alpha\pi}$

if $c_2 \neq 0$, $0 = -e^{\alpha\pi} + e^{-\alpha\pi} \Rightarrow e^{\alpha\pi} = e^{-\alpha\pi}$

$\Rightarrow e^{2\alpha\pi} = 1$

$2\alpha\pi = 0$

$\alpha = 0$ impossible

so $c_2 = 0$, and $c_1 = -c_2 = 0$ as well.

$\lambda < 0$ is never an eigenvalue, only solution is $y = 0$.

Summary for $y'' + \lambda y = 0 \quad y(0) = 0 = y(\pi)$

Eigenvalues: $\lambda_n = n^2$ for $n = 1, 2, 3, 4, \dots$

$\lambda_1 = 1, \quad \lambda_2 = 4, \quad \lambda_3 = 9, \quad \lambda_4 = 16, \dots$

Eigen functions.

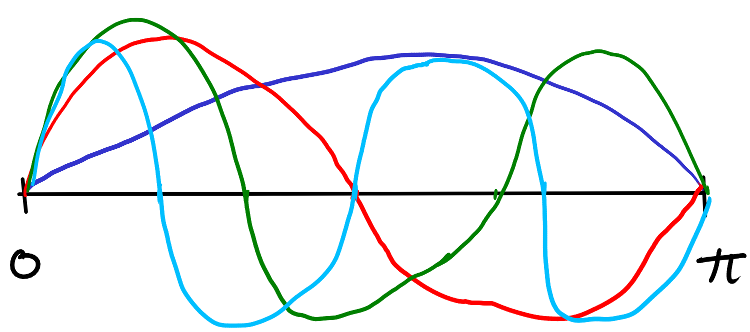
$\lambda_1 = 1 \qquad y_1(x) = c \sin x$

$\lambda_2 = 4 \qquad y_2(x) = c \sin 2x$

$\lambda_3 = 9 \qquad y_3(x) = c \sin 3x$

$\vdots$

$\lambda_n = n^2 \qquad y_n(x) = c \sin nx$



Can also consider: $y'' + \lambda y = 0 \quad y(0) = 0 = y(L)$

look for $\lambda = \mu^2$ an eigenvalue need $\sin \mu L = 0$

hence $\mu L = n\pi \Rightarrow \mu = \frac{n\pi}{L}$

Eigenvalue $\lambda_n = \left(\frac{n\pi}{L}\right)^2$

eigenfunctions $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $y_n(x) = c\sin\left(\frac{n\pi}{L}x\right)$