Review & Problem solving

Eigenvalues and eigenvectors. for 2x2

$$A = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$$ Find eigenvalues and eigenvectors

Eigenvalues $\det(A - rI) = \det \begin{pmatrix} -1-r & -4 \\ 1 & -1-r \end{pmatrix}$

$$= (-1-r)(-1-r) - 1\cdot(-4)$$

$$= 1 + 2r + r^2 + 4 = r^2 + \underset{\substack{\uparrow \\ 2 = -\text{trace } A}}{2r} + \underset{\substack{\uparrow \\ \det A = 5}}{5}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{trace} = a + d$$

$$\det \begin{pmatrix} a-r & b \\ c & d-r \end{pmatrix} = (a-r)(d-r) - bc$$

$$= r^2 - (a+d)r + (ad - bc)$$

roots of $r^2 + 2r + 5$ $\quad r = \dfrac{-2 \pm \sqrt{4-20}}{2}$

$$= -1 \pm 2i$$

Find eigenvector for $-1 + 2i$

$$\begin{pmatrix} -1-(-1+2i) & -4 \\ 1 & -1-(-1+2i) \end{pmatrix} = \begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix}$$

$$\begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix}\begin{pmatrix} 4 \\ -2i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

look at first row $(-2i \;\; -4)$ write it as a column

$$\begin{pmatrix} -2i \\ -4 \end{pmatrix} \xrightarrow{\text{swap}} \begin{pmatrix} -4 \\ -2i \end{pmatrix} \xrightarrow[\text{one sign}]{\text{change}} \begin{pmatrix} 4 \\ -2i \end{pmatrix}$$

Do this with either row, and other row will work automatically, unless you made a mistake with the eigenvalues.

$r_1 = -1+2i \qquad \vec{v}_1 = \begin{pmatrix} 4 \\ -2i \end{pmatrix}$

other eigenstuff is the complex conjugate

$r_2 = -1-2i \qquad \vec{v}_2 = \begin{pmatrix} 4 \\ 2i \end{pmatrix}$

Complex solutions: $e^{rt}\vec{v}$ $\qquad \begin{cases} e^{(-1+2i)t}\begin{pmatrix} 4 \\ -2i \end{pmatrix} \\ e^{(-1-2i)t}\begin{pmatrix} 4 \\ 2i \end{pmatrix} \end{cases}$

To get real solutions, take the real and imaginary parts of either of the complex solutions

Write $e^{(-1+2i)t}\begin{pmatrix}4\\-2i\end{pmatrix}$ in terms of real and imag. parts

$$\underset{e^{-t}e^{2it}}{e^{(-1+2i)t}} = e^{-t}\cos 2t + i e^{-t}\sin 2t$$

solutions $\begin{pmatrix}4(e^{-t}\cos 2t + i e^{-t}\sin 2t)\\ -2i(e^{-t}\cos 2t + i e^{-t}\sin 2t)\end{pmatrix}$

$$= \begin{pmatrix}4e^{-t}\cos 2t + i\, 4e^{-t}\sin 2t\\ -2i\, e^{-t}\cos 2t + 2e^{-t}\sin 2t\end{pmatrix}$$

$$= \begin{pmatrix}4e^{-t}\cos 2t + i\, 4e^{-t}\sin 2t\\ 2e^{-t}\sin 2t + i(-2)e^{-t}\cos 2t\end{pmatrix}$$

$$= \underset{\vec{u}}{\underset{\shortparallel}{\begin{pmatrix}4e^{-t}\cos 2t\\ 2e^{-t}\sin 2t\end{pmatrix}}} + i\underset{\vec{v}}{\underset{\shortparallel}{\begin{pmatrix}4e^{-t}\sin 2t\\ -2e^{-t}\cos 2t\end{pmatrix}}}$$

general solution: $\vec{X} = C_1\vec{u} + C_2\vec{v}$

Ex $y^{(4)} - 4y''' + 4y'' = 0$

Characteristic equation: $r^4 - 4r^3 + 4r^2 = 0$

Using initial condition: $y(0) = y'(0) = y''(0) = y'''(0) = 0$

$\mathcal{L}\{y^{(4)} - 4y''' + 4y''\} = (s^4 - 4s^3 + 4s^2)Y(s)$

$r^2(r^2 - 4r + 4) = r^2(r-2)^2 = 0$

$r = 0, 0, 2, 2$

$$\underset{\substack{\| \\ 1}}{e^{0t}} \quad \underset{\substack{\| \\ t}}{te^{0t}} \quad e^{2t} \quad te^{2t}$$

general sol $y = c_1 + c_2 t + c_3 e^{2t} + c_4 te^{2t}$

$y'' - xy' - y = 0$ at $x_0 = 0$

Seek a power series solution, find the recurrence relation.

relationship between the coefficients

$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n x^n$

since $x_0 = 0$.

$y' = \sum_{n=0}^{\infty} a_n \cdot n \cdot x^{n-1}$ (Don't shift immediately)

$a_{n+2} =$ blah a_{n+1} + blah a_n

$$xy' = x\sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_n \cdot n \cdot x^n$$

At what n-value does the sum start?

$$\sum_{n=0}^{\infty} a_n \cdot n \cdot x^{n-1} = \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}$$

Works because $n=0$ term is $a_0 \cdot 0 \cdot x^{-1} = 0$

$\frac{d}{dx}(x^n) = nx^{n-1}$ if $n=0$ $\frac{d}{dx}(1) = 0.$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

because $n=0$ and $n=1$ terms are 0.

shift by 2 $= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n$

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

$y'' = 2a_2 + 6a_3 x + \cdots$

Putting it together $\sum (n+2)(n+1)a_{n+2}x^n$

$\boxed{y'' - xy' - y = 0}$ $- \sum n a_n x^n - \sum a_n x^n = 0$

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}-na_n-a_n\right]x^n=0$$

$$(n+2)(n+1)a_{n+2}-na_n-a_n=0$$

Solve for higher interms of lower a_{n+2} in terms of a_n

$$\left.a_{n+2}=\frac{a_n}{n+2}\right]$$ Recurrence Relation.

Start with $a_0=1$, $a_1=7$,

$a_2=\frac{1}{2}$ $a_3=\frac{7}{3}$ $a_4=\frac{1}{8}$ $a_5=\frac{7}{15}$

$$\boxed{y=1+7x+\frac{1}{2}x^2+\frac{7}{3}x^3+\frac{1}{8}x^4+\frac{7}{15}x^5+\cdots}$$

Find the inverse laplace transform of

$$F(s)=\frac{(s-2)e^{-s}}{s^2-4s+3}=\frac{(s-2)e^{-s}}{(s-2)^2-1}$$

$$=\frac{(s-2)e^{-s}}{(s-1)(s-3)}=e^{-s}\left(\frac{s-2}{(s-1)(s-3)}\right)$$

$$\mathcal{L}\{u_c(t)\,f(t-c)\} = e^{-cs}\,\mathcal{L}\{f(t)\}$$

step function — shift — exponential — NO shift

$c = 1$ in our problem: $\mathcal{L}\{f(t)\} = \dfrac{s-2}{(s-1)(s-3)}$

Take inverse transform of $\dfrac{s-2}{(s-1)(s-3)}$ Partial fractions

$$\frac{s-2}{(s-1)(s-3)} = \frac{A}{s-1} + \frac{B}{s-3}$$

Clear denominator: $s-2 = A(s-3) + B(s-1)$

plug in $s=3$: $3-2 = A(3-3) + B(3-1)$

$1 = 2B \qquad B = 1/2$

plug in $s=1$: $1-2 = A(-2) + 0$

$-1 = -2A \qquad A = 1/2$

$$\mathcal{L}^{-1}\left\{\frac{1/2}{s-1} + \frac{1/2}{s-3}\right\} = \frac{1}{2}e^{t} + \frac{1}{2}e^{3t} = f(t)$$

$$e^{-s}\mathcal{L}\{f(t)\} \xrightarrow{\mathcal{L}^{-1}} u_1(t)\,f(t-1)$$

$$= u_1(t)\left[\frac{1}{2}e^{(t-1)} + \frac{1}{2}e^{3(t-1)}\right]$$

$$= \begin{cases} 0, & t < 1 \\ \frac{1}{2}e^{(t-1)} + \frac{1}{2}e^{3(t-1)}, & t \geq 1 \end{cases}$$

$$F(s) = \frac{2s+2}{s^2+2s+5}$$

$b^2 - 4ac = -16 < 0$ denominator can't be factored

Other idea complete the square

$$(s+1)^2 + 4 = s^2+2s+5$$

look at

$$\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$$

$$\frac{2s+2}{(s+1)^2+4} = 2\left(\frac{s+1}{(s+1)^2+2^2}\right)$$

$$\rightarrow 2e^{-t}\cos 2t$$