Types of DEs / First Order Linear ODE

Quiz tomorrow from HW $\quad \frac{dy}{dt} = ay - b$

Solutions to HW in back of book.

My OH $\quad$ T : 3:30 – 5:00
Th : 11:00 – 12:30

Types of Differential Equations (DE)

→ Ordinary vs Partial ⇔ # of independent variables

Ordinary → 1 indep variable $\quad t$

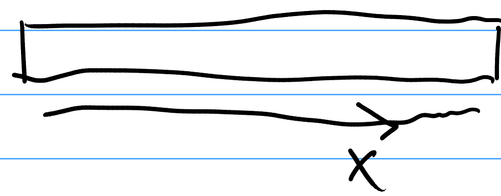
$$y(t) \qquad y' = f(t,y)$$

$$y'' = (y')^2 + 2y + t$$

Partial DE → at least 2 independent vars.

$$u = u(t,x) \qquad \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$ heat equation

heat conduction in a rod.



Systems $\to$ more than 1 dependent verteble
more than 1 equation.

$x(t)$, $y(t)$ functions

$$\frac{dx}{dt} = ax - \alpha xy$$
$$\frac{dy}{dt} = -cy + \gamma xy$$

Order of a DE $\to$ highest order of derivatrie that appears

y' but not y'', y''', etc $\to$ first order

y'' but not y''', ... $\to$ second order

In this course we assume the DE has been solved for the highest order derivatives

general form of n th order Ordinary Diff Eq.

$$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)})$$

$$(y')^2 = y \implies y' = \sqrt{y}$$
$$\text{or } y' = -\sqrt{y}$$

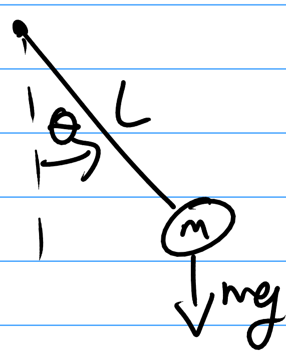
Important Distinction Linear vs Nonlinear.

Linear means the unknown function and it's derivatives appear in the equation to the first power only.

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_{n-1}(t)\,y' + a_n(t)\,y = g(t)$$

Analogous to $\quad y = mx + b$

Not linear



$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

Reason for Linear: principle of superposition

$$a(t)\,y' + b(t)\,y = 0$$ (homogeneous equation because RHS = 0)

Then if $y_1(t)$ and $y_2(t)$ are two solutions then $y_1(t) + y_2(t)$ is also a solution.

Given an equation

- Find solutions → Do they exist?
- When they do exist, how many are there?
- Can you actually write them down?

$u = u(x,y)$

$$\begin{cases} \dfrac{\partial u}{\partial x} = y \\ \dfrac{\partial u}{\partial y} = 2x \end{cases}$$

No solution.

$$\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial}{\partial y} y = 1$$

$$\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = \frac{\partial}{\partial x} 2x = 2$$

First Order Linear Ordinary Diff Eq

$$a(t)\, y' + b(t)\, y = c(t)$$ general form

Divide by $a(t)$: $y' + \dfrac{b(t)}{a(t)} y = \dfrac{c(t)}{a(t)}$

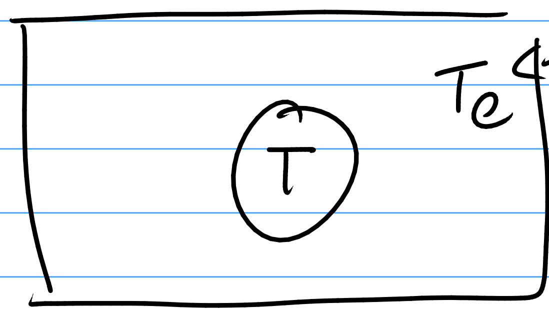
STANDARD LINEAR FORM

$$y' + p(t)\, y = q(t)$$

If $q(t) = 0$, the equation is called homogeneous.

- Can be solved analytically.
- Comes up in many models

Newton's law of cooling:



environmental ambient temperature.

$$\frac{dT}{dt} = k(T_e - T)$$

k conductivity
$k > 0$

$$\frac{dT}{dt} = -kT + kT_e$$

(if k and T_e are const, get $\frac{dy}{dt} = ay - b$)

What if $T_e = T_e(t)$ is not constant
(eg. changing temperature of environment)

Solving $y' + p(t)y = q(t)$

Trick: Integrating factor

multiply <u>both</u> sides of equation by a function $u(t)$ so that it simplifies.

Try it: $y' + p(t)y = q(t)$

$$u(t)y' + u(t)p(t)y = u(t)q(t)$$

$(u(t)y)' = u(t)y' + u'(t)\,y$

Difference is u' vs $u \cdot p$

Look for u such that $u' = u \cdot p$

$$\frac{du}{dt} = up \Rightarrow \int \frac{du}{u} = \int p\,dt$$

$$\Rightarrow \ln u = \int p\,dt \qquad u = e^{\int p\,dt}$$

Note: there's no constant of integration because we just need one function $u(t)$

Can also assume u is positive.

check $\frac{du}{dt} = e^{\int p\,dt} \frac{d}{dt} \int p\,dt = up$

For $y' + p(t)y = q(t)$
The integrating factor is $e^{\int p\,dt}$

Solution method

0. Put in standard form $y' + py = q$
1. Find integrating factor $e^{\int p\,dt}$
2. Multiply both sides by $e^{\int p\,dt}$
3. Combine terms on left hand side and integrate

Ex $ty' - y = t^3$ Consider $t > 0$

0 Standard form $y' - \frac{1}{t} y = t^2$

1 Integrating factor

no abs val

$$\int p\,dt = \int -\frac{1}{t}\,dt = -\ln t$$

$$u = e^{-\ln t} = \left(e^{\ln t}\right)^{-1} = (t)^{-1} = \frac{1}{t}$$

2 Multiply both sides by $u = \frac{1}{t}$

$$\frac{1}{t} y' - \frac{1}{t^2} y = t$$

$$\left(\frac{1}{t} y\right)' = \frac{1}{t} y' - \frac{1}{t^2} y = t$$

combine

product rule.

Now we have

$\left(\frac{1}{t}y\right)' = t$ ← integrate both sides with respect to t

$$\frac{1}{t}y = \int t\,dt + c = \frac{t^2}{2} + c$$

$$y = \frac{t^3}{2} + ct$$

$$\frac{dT}{dt} = k(T_e - T) \qquad T_e = T_e(t)$$

Standard form: $\frac{dT}{dt} + kT = kT_e(t)$

$p(t) = k$

I.F. $e^{\int k\,dt} = e^{kt}$

$$e^{kt}\frac{dT}{dt} + ke^{kt}T = ke^{kt}T_e(t)$$

$$\frac{d}{dt}\left(e^{kt}T\right) = ke^{kt}T_e(t)$$

$$e^{kt}T = \int_0^t ke^{kt'}T_e(t')\,dt' + C$$

$$T = e^{-kt}\left[\int_0^t ke^{kt'}T_e(t')\,dt' + C\right]$$