Exam next Tuesday 11/13 usual time and place

chapters 4, 5*, 6*, 7

Practice Problems posted on website ←

Extra Office hours : Friday. 2:00 pm - until we get tired.

Linear systems via eigenvalues and eigenvectors

2x2 system $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

① $\vec{x}'(t) = A\vec{x}(t)$ $\qquad$ ② $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

③ $\begin{cases} x_1' = ax_1 + bx_2 \\ x_2' = cx_1 + dx_2 \end{cases}$

①, ②, ③ are just different notations.

2x2 First order linear homogeneous system.

→ To specify a unique solution, need initial condition.

initial values of x_1 and x_2 i.e. need to know $\vec{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$

→ Because the initial condition involve specifying 2 numbers, the general solution must have 2 undetermined constants in it.

→ A fundamental set of solutions consists of 2 linearly independent solutions.

$$x^{(1)}(t) \qquad x^{(2)}(t)$$

general $\quad x(t) = \underbrace{c_1 x^{(1)}(t) + c_2}_{\text{constants}} x^{(2)}(t)$

use initial condition to solve for constants.

<u>Example</u>: $\quad \vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x} \quad$ Find general solution

Try $\vec{x}(t) = \vec{v} e^{rt}$ where $\vec{v}$ is some constant vector.

When does it solve $\quad \vec{x}'(t) = \frac{d}{dt}(\vec{v} e^{rt}) = \vec{v} \frac{d}{dt}(e^{rt})$

$$= \vec{v} r e^{rt}$$

what is $\quad A\vec{x} = \underbrace{A\vec{v}}_{\text{vector}} e^{rt}$

Set equal: $\quad \vec{v} r e^{rt} = A\vec{v} e^{rt} \iff \vec{v} \cdot r = A\vec{v}$

$\vec{v}r = A\vec{v} \xrightarrow{\text{rewrite}} A\vec{v} = r\vec{v}$ Eigenvalue Eigenvector equation

r must be an eigenvalue of A, and $\vec{v}$ must be an eigenvector of A associated to the eigenvalue r.

characteristic equation

To find eigenvalues, solve $\det(A - rI) = 0$

$$\det\left(\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - r\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\begin{pmatrix} 1-r & 1 \\ 4 & 1-r \end{pmatrix}$$

$$= (1-r)^2 - 4 = 1 - 2r + r^2 - 4 = r^2 - 2r - 3$$

Characteristic equation $r^2 - 2r - 3 = 0$

solutions $r_1 = 3 \quad r_2 = -1$ ← these are the possible r's.

Find eigen vector for $r_1 = 3$

$$(A - 3I)\vec{v} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0, \quad \vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

eigenvector for $r_2 = -1$

$$(A + I)\vec{v} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \qquad \vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Solutions

Fundamental set

$$x^{(1)}(t) = e^{r_1 t}\vec{v}^{(1)} = e^{3t}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}e^{3t}\\2e^{3t}\end{pmatrix}$$

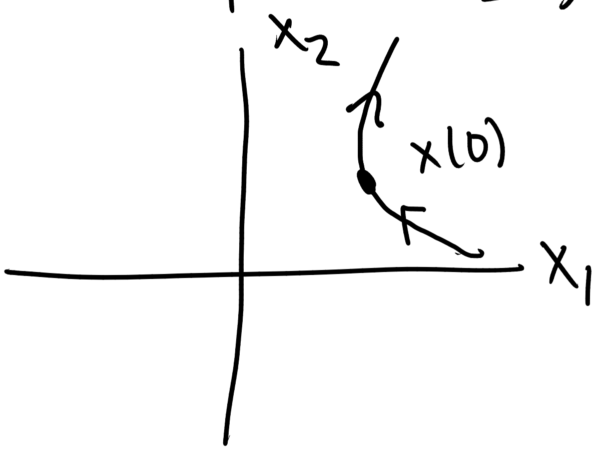
$$x^{(2)}(t) = e^{r_2 t}\vec{v}^{(2)} = e^{-t}\begin{pmatrix}1\\-2\end{pmatrix} = \begin{pmatrix}e^{-t}\\-2e^{-t}\end{pmatrix}$$

General solution: $x(t) = c_1 e^{3t}\begin{pmatrix}1\\2\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\-2\end{pmatrix}$

$$= \begin{pmatrix}c_1 e^{3t} + c_2 e^{-t}\\ 2c_1 e^{3t} - 2c_2 e^{-t}\end{pmatrix} \qquad \begin{cases} x_1 = c_1 e^{3t} + c_2 e^{-t} \\ x_2 = 2c_1 e^{-t} - 2c_2 e^{-t}\end{cases}$$

Geometry is visible in the "phase plot"

Plot x_1 and x_2, but don't plot t.

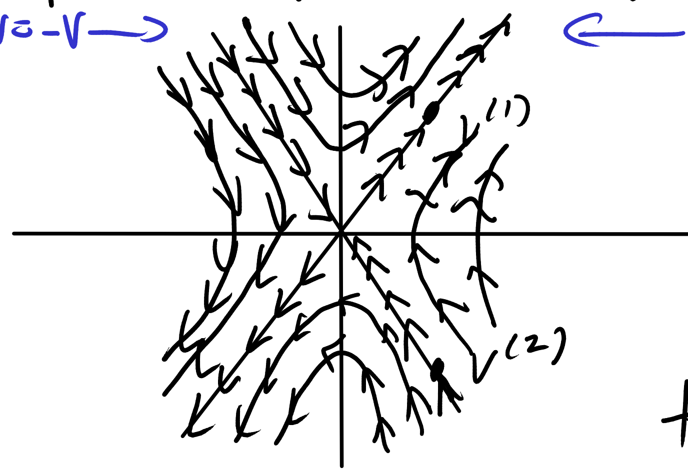


Path traced out by the solution.

In our example $\quad r_1 = 3 \quad v^{(1)} = \begin{pmatrix}1\\2\end{pmatrix} \qquad r_2 = -1 \quad v^{(2)} = \begin{pmatrix}1\\-2\end{pmatrix}$

here $Av = -v$ ⟶ ⟵ here $Av = 3v$

$x' = Ax$



$\boxed{\text{Saddle point}}$

eigenvectors ⟷ trajectories in/out of $(0,0)$

Example: $x' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} x$ $\quad A = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}$

Find general solution.

step 1 find eigenvalues $\det(A - rI) = 0$

$$\det \begin{pmatrix} -3-r & \sqrt{2} \\ \sqrt{2} & -2-r \end{pmatrix} = (-3-r)(-2-r) - 2$$
$$= r^2 + 5r + 4$$
$$= (r+1)(r+4)$$

eigenvalue $r_1 = -1 \quad r_2 = -4$

Step 2 find eigenvectors

$r_1 = -1 \quad : 0 = (A + 1I)\vec{v} = \begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1 \end{pmatrix} = 0$

$\left(\text{or} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \right)$

$r_2 = -4 : (A + 4I)\vec{v} = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} = 0$

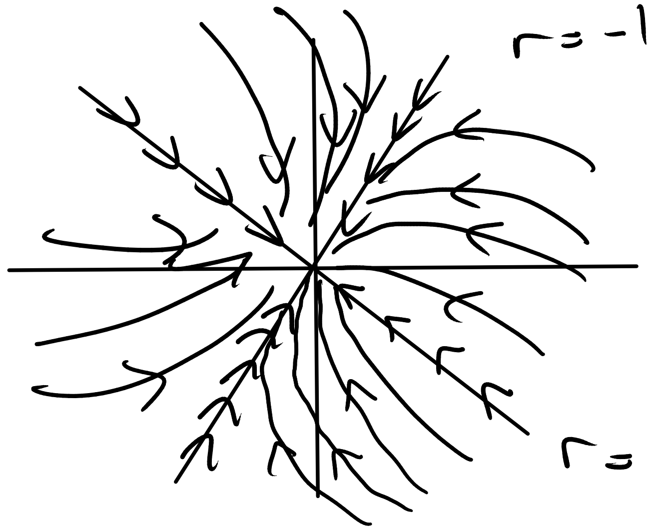
$\left(\text{or} \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} \right)$

Step 3 write solutions

$x^{(1)}(t) = e^{-t} \begin{pmatrix} 1/\sqrt{2} \\ 1 \end{pmatrix}$ $\quad x^{(2)}(t) = e^{-4t} \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$

general solution $x(t) = c_1 e^{-t} \begin{pmatrix} 1/\sqrt{2} \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$

Plase plot



negative eigenvalues $\Rightarrow$ flow in

everything flows in $\Rightarrow$ "node"

Complex eigenvalues:

<u>Example</u>: $x' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x$

Step 1 find eigenvalues

$$\det(A - rI) = \det\begin{pmatrix} -r & -1 \\ 1 & -r \end{pmatrix} = r^2 + 1 = 0$$

$$r_1 = i \quad r_2 = -i$$

eigenvectors

$r_1 = i \qquad \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = 0 \qquad x^{(1)}(t) = e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix}$

$r_2 = -i \qquad \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}\begin{pmatrix} -i \\ 1 \end{pmatrix} = 0 \qquad x^{(2)}(t) = e^{-it}\begin{pmatrix} -i \\ 1 \end{pmatrix}$

$e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix}$ and $e^{-it}\begin{pmatrix} -i \\ 1 \end{pmatrix}$ are fundumental set of complex-valued solutions

eigenvalues are complex conjugates, and so are eigenvector

Step 4:

To get real solutions, take real and imaginary parts of any complex solution

$$e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix} = (\cos t + i \sin t)\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} i\cos t - \sin t \\ \cos t + i \sin t \end{pmatrix}$$

$$= \begin{pmatrix} -\sin t + i\cos t \\ \cos t + i \sin t \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

these are a fundamental set of real solutions.

General solution:

$$x(t) = c_1\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$