First order Systems of linear equations

System of differential equation: more than one dependent variable / unknown function.

x_1 and x_2 are functions of time t

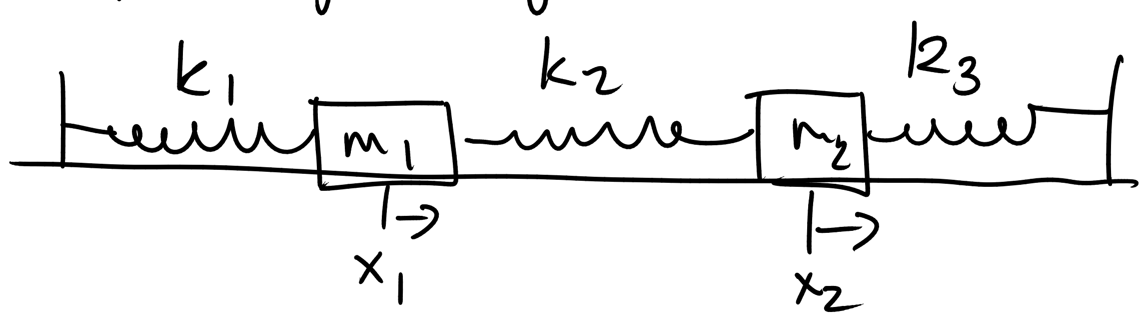
$$\left\{\begin{aligned} \frac{dx_1}{dt} &= x_1(1-x_1-x_2) \\ \frac{dx_2}{dt} &= x_2(1-x_1-x_2) \end{aligned}\right.$$

Nonlinear system.

"coupled" x_1 and x_2 appear in both equations.

E.g. x_1 = population of owls x_2 = pop of mice

Multiple degrees of freedom



$$m_1 \frac{d^2x_1}{dt^2} = k_2(x_2-x_1) - k_1x_1$$

$$m_2 \frac{d^2x_2}{dt^2} = -k_3x_2 - k_2(x_2-x_1)$$

Principle : Higherorder differential equations can be translated into first order systems.

E.g. $u'' + 0.5u' + 2u = 0 \quad (*)$

introduce new variables to represent the derivatives

$$\boxed{\begin{aligned} x_1 &= u \\ x_2 &= u' \end{aligned}}$$ ← Define x_1 and x_2 this way

write equation $(*)$ in terms of x_1, x_2 and x_1', x_2'

$u'' = x_2' \quad u' = x_2 \quad u = x_1$

$$\left.\begin{aligned} (*) \quad & x_2' + 0.5x_2 + 2x_1 = 0 \\ & x_2 = x_1' \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} x_1' &= x_2 \\ x_2' &= -2x_1 - 0.5x_2 \end{aligned}\right.$$

E.g. $y^{(4)} - y = 0$ $\quad y^{(4)} = y$ $\quad$ write as first order system

variables = order of the original equation = 4

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \\ x_3 &= y'' \\ x_4 &= y''' \end{aligned} \qquad \left\{\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ x_4' &= x_1 \end{aligned}\right\}$$

$$y'' + p(t)y' + q(t)y$$

General first order linear system: unknown functions $x_1, x_2, \ldots, x_n$

$$x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + g_1(t)$$

$$x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + g_2(t)$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$x_n' = \underbrace{p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n}_{\text{homogeneous part}} + \underbrace{g_n(t)}_{\text{nonhomogeneous (forcing) term}}$$

<u>Ex:</u> $\begin{cases} x_1' = x_1 + x_2 \\ x_2' = 4x_1 + x_2 \end{cases}$

<u>Matrices:</u> 2×2 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

3×3 $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

column vectors $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

$$(m\times n) \text{ times } (n\times p) \text{ matrix} = \begin{array}{ccc} \text{rows} & & \text{columns} \\ \downarrow & & \downarrow \\ m & \times & p \end{array} \text{ matrix}$$

matrix $\times$ column = column

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix}$$

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{array}\right\} \Longleftrightarrow A\vec{x} = \vec{b}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Given a square $(n\times n)$ matrix A, we consider 2 equations

Homogeneous equation $\quad A\vec{x} = 0 \qquad 0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Nonhomogeneous equation. $\quad A\vec{x} = \vec{b}$

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

A can either be singular or nonsingular

A nonsingular $\Longleftrightarrow \det A \neq 0 \Longleftrightarrow$ columns of A are linearly independent

$\Longleftrightarrow A\vec{x} = 0$ has $\vec{x} = 0$ as the only solution

$\iff$ $A\vec{x}=\vec{b}$ has a unique solution for every $\vec{b}$

$\iff$ an inverse matrix, A^{-1} exists
$(AA^{-1} = A^{-1}A = I)$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \det A = ad - bc$$

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

check
$$AA^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Take a column $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ what is $I\vec{x}$?

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad I\vec{x} = \vec{x}$$

Eigenvalues and Eigenvectors: Let A be a square $(n \times n)$ matrix

The Eigenvalue-Eigenvector equation is

$$A\vec{x} = \lambda\vec{x}$$

where $\vec{x}$ is an unknown vector and λ is an unknown number.

if λ and $\vec{x}$ solve, then λ is called an eigenvalue and $\vec{x}$ is called an eigenvector for the eigenvalue λ.

($\vec{x} = 0$ λ = anything solves always so I don't want that).

Another form $\lambda \vec{x} = \lambda I \vec{x}$

so $A\vec{x} = \lambda \vec{x} \Leftrightarrow A\vec{x} = \lambda I \vec{x} \Leftrightarrow A\vec{x} - \lambda I \vec{x} = 0$

$\Leftrightarrow (A - \lambda I)\vec{x} = 0$

If a nonzero solution $\vec{x}$ is to exist, then

$A - \lambda I$ must be singular, and $\det(A - \lambda I) = 0$

Eg $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ $A\vec{x} = \lambda \vec{x}$

What are the possible λ's? need $\det(A - \lambda I)$

$$A - \lambda I = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix}$$

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4 = 1 - 2\lambda + \lambda^2 - 4$$

$$= \lambda^2 - 2\lambda - 3 = 0$$

Eigenvalues of A $\lambda_1 = 3$, $\lambda_2 = -1$

$$(\lambda - 3)(\lambda + 1) = 0$$

Eigenvector for $\lambda_1 = 3$ $\quad (A - 3I)\vec{x} = 0$

$$A - 3I = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} x_1 = 1 \\ x_2 = 2 \end{matrix}$$

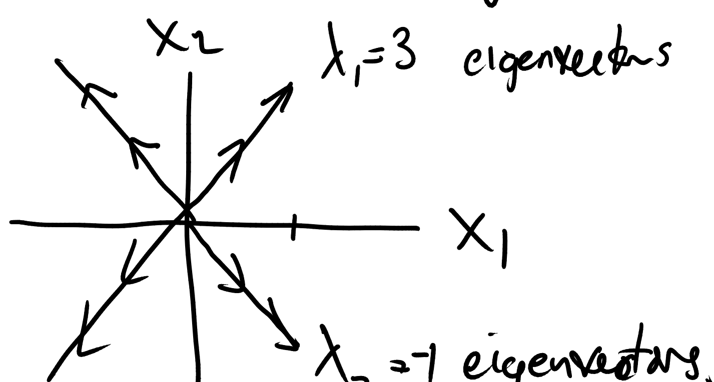
$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

is an eigenvector for $\lambda_1 = 3$ of A

also $\begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

or any constant multiple of $\vec{x}_1$ $\quad c\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

plot in 2d



Eigenvectors for $\lambda_2 = -1$

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \vec{x}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

or $\vec{x}_1 = c\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

2x2 $\quad A\vec{x} = \lambda\vec{x}$

3 cases : 1st case : eigenvalues are real and distinct

↳ 2 eigenvalues and 2 eigenvectors

2nd case : eigenvalues are complex conjugates

↳ 2 complex eigenvalues and 2 complex eigenvectors

3rd case : eigenvalue is a repeated root of

$$\det(A - \lambda I) = 0$$

1 eigenvalue one or two eigenvectors