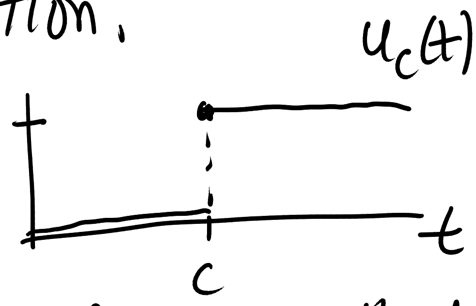
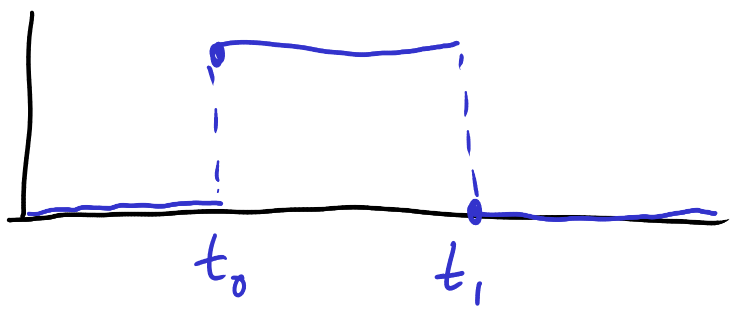
Impulse Functions & Convolution.

Last time: step functions



Using $u_c(t)$, can look at Forcing functions that turn ON and OFF.



$$g(t) = u_{t_0}(t) - u_{t_1}(t)$$

$$mu'' + \gamma u' = mg + F(t)$$

An <u>impulse</u> means the force is applied instantaneously

If $g(t)$ is any forcing function then the <u>impulse</u> of $g(t)$ is

$$I = \int_{-\infty}^{\infty} g(t)\,dt = \int_{t_0}^{t_1} g(t)\,dt$$

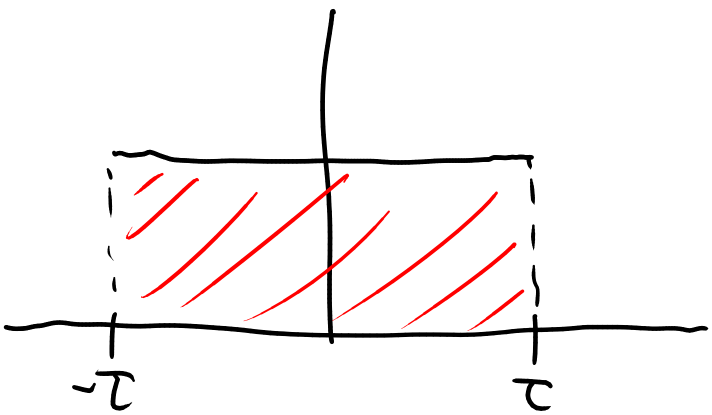
if $g(t) = 0$ for $t < t_0$, $t > t_1$

2nd law $\quad F = \dfrac{dp}{dt}$

$$I = \Delta(\text{momentum})$$

Want to consider the limit where the impulse = change in momentum is fixed, but the time interval over which the force is applied goes to 0.

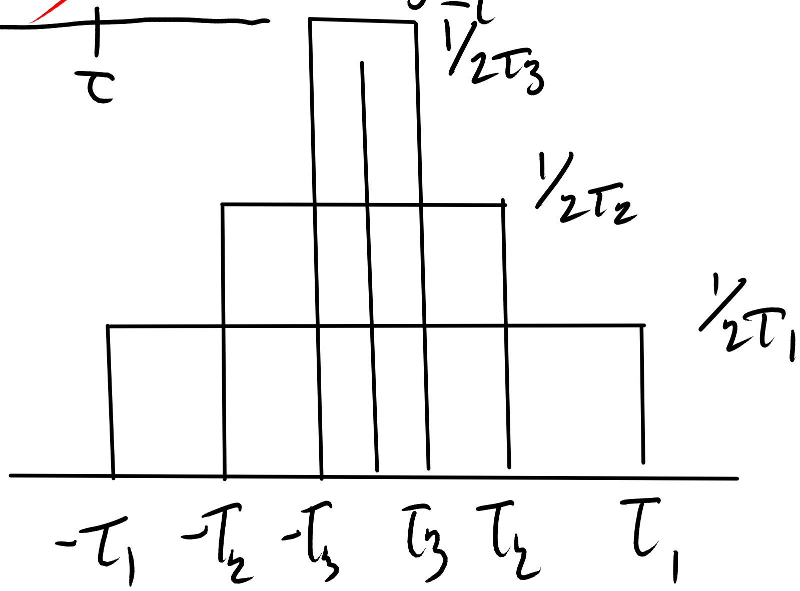
$$d_\tau(t) = \begin{cases} 1/2\tau & -\tau < t < \tau \\ 0 & \text{otherwise} \end{cases}$$



$$I = \int_{-\tau}^{\tau} d_\tau(t)\,dt$$

$$= \int_{-\tau}^{\tau} \frac{1}{2\tau}\,dt = \frac{1}{2\tau}\,2\tau = 1$$

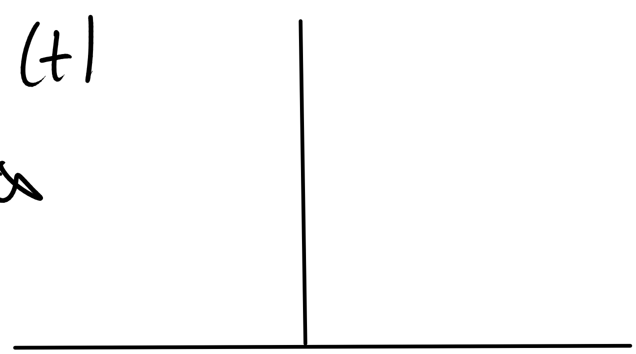
As $\tau \to 0$:



<u>"Definition"</u>

$$\delta(t) = \lim_{\tau \to 0} d_\tau(t)$$

Graph is "rectangle with height ∞ and width zero".



Properties: $\cdot\ \delta(t) = 0$ unless $t=0$

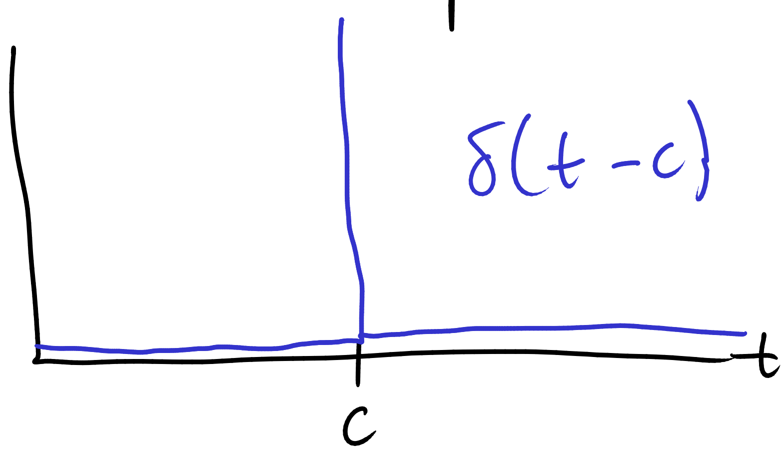
$\cdot\ \displaystyle\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

Thinking of $\delta(t)$ as a function is a useful fiction (strictly speaking $\delta(t)$ is a "distribution" or "generalized function")

Called Dirac Delta Function or unit impulse function

In physical terms it represents a unit impulse delivered instantaneously.

We can have the impulse "hit" at time $t=c$



KEY PROPERTY OF $\delta(t-c)$ is

if $f(t)$ is a continuous function

then $\displaystyle\int_{-\infty}^{\infty} \delta(t-c)\,f(t)\,dt = f(c)$

"Proof"

$$\int_{-\infty}^{\infty} \delta(t-c) f(t)\,dt = \lim_{\tau \to 0} \int_{-\infty}^{\infty} d_\tau(t-c) f(t)\,dt$$

$$= \lim_{\tau \to 0} \frac{1}{2\tau} \int_{c-\tau}^{c+\tau} f(t)\,dt \underset{FTC}{\approx} \frac{1}{2\tau} 2\tau f(c) = f(c)$$

$$\mathcal{L}\{\delta(t-c)\} = \int_0^\infty \delta(t-c) e^{-st}\,dt = e^{-sc}$$

$c > 0$

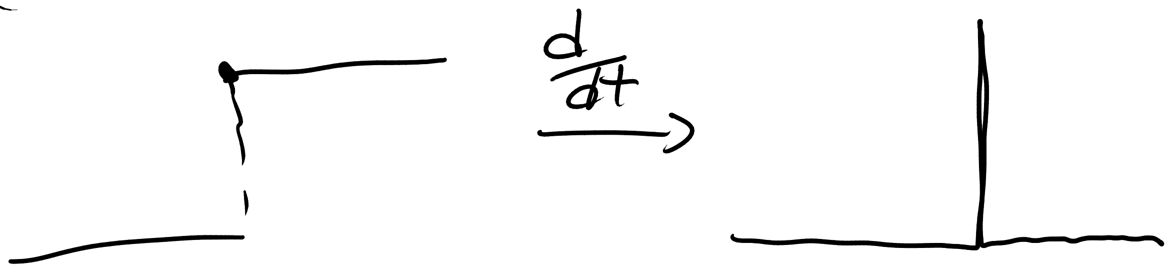
If $c = 0$: $\mathcal{L}\{\delta(t)\} = 1$

Compare $\mathcal{L}\{u_c(t)\} = \dfrac{e^{-cs}}{s}$

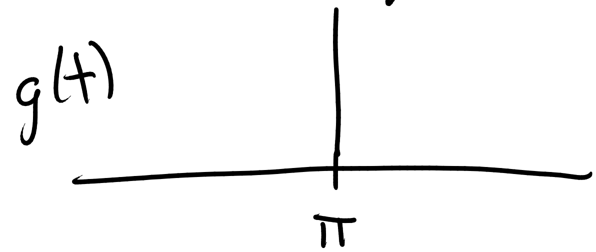
Note $\mathcal{L}\{\delta(t-c)\} = s\,\mathcal{L}\{u_c(t)\}$

$$\delta(t-c) = \frac{d}{dt}\left(u_c(t)\right)$$

(strictly $u_c(t)$ isn't differentiable a c and $\delta(t-c)$ isn't even an honest function)



Solve $y'' + 4y = \delta(t-\pi) \quad y(0) = 0, \; y'(0) = 0$



$$Y(s) = \mathcal{L}\{y(t)\}$$

$$\mathcal{L}\{y''(t)\} = s^2 Y(s) - s y(0) - y'(0)$$

$$= s^2 Y(s)$$

$$\mathcal{L}\{\delta(t-\pi)\} = e^{-\pi s}$$

$$y'' + 4y = \delta(t-\pi)$$

$$s^2 Y(s) + 4Y(s) = e^{-\pi s} \longrightarrow Y(s) = e^{-\pi s} \frac{1}{s^2+4}$$

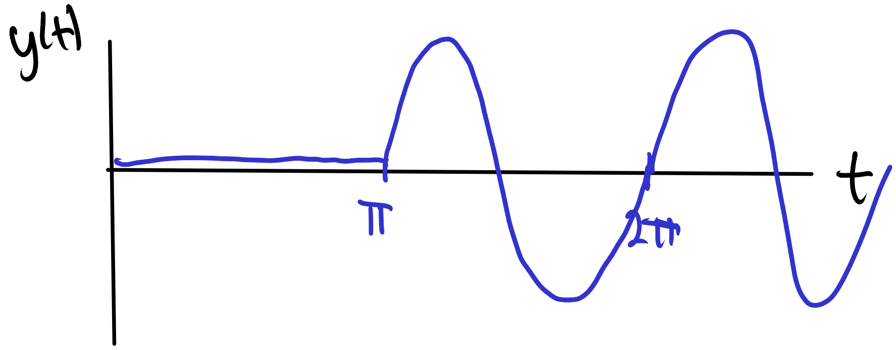
Recall: $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} F(s)$

$$Y(s) = e^{-\pi s} \frac{1}{s^2+4} = e^{-\pi s} H(s) \text{ where } H(s) = \frac{1}{s^2+4}$$

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{1}{2} \sin 2t$$

$$y(t) = u_\pi(t)\, h(t-\pi) = u_\pi(t) \cdot \frac{1}{2} \cdot \sin[2(t-\pi)]$$

$$= u_\pi(t) \cdot \frac{1}{2} \cdot \sin(2t)$$



Kick it and it starts working.

Convolution : what is $\mathcal{L}^{-1}\{F(s)\,G(s)\}$

eg. $\frac{1}{s}\cdot\frac{1}{(s^2+4)}$ or $e^{-5s}\frac{1}{s^2+2s+2}$ etc.

$\mathcal{L}\{f(t)\,g(t)\} = F(s)\,G(s)$ very wrong.

Q can we find $h(t)$ in terms of $f(t)$ and $g(t)$

so that $\mathcal{L}\{h(t)\} = F(s)\,G(s) = \mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\}$

$$h(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau = (f*g)(t)$$

is called the convolution of f and g.

To ~~proof~~

$$F(s)\,G(s) = \left(\int_0^\infty f(t)e^{-st}dt\right)\left(\int_0^\infty g(t)e^{-st}dt\right)$$

→ combine these into a double integral, and do a change of variables to get

$$\int_0^\infty\left(\int_0^t f(t-\tau)\,g(\tau)\,d\tau\right)e^{-st}dt$$

Can use this as alternative to partial fractions

$$H(s) = \frac{1}{s^2(s-5)} = \underset{\underset{F(s)}{\parallel}}{\frac{1}{s^2}} \cdot \underset{\underset{G(s)}{\parallel}}{\frac{1}{s-5}}$$

$f(t) = t \quad g(t) = e^{5t}$

$$h(t) = \mathcal{L}^{-1}(H(s)) = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

$$= \int_0^t (t-\tau)e^{5\tau}\,d\tau$$

$$= t\int_0^t e^{5\tau}\,d\tau - \int_0^t \tau e^{5\tau}\,d\tau = \frac{1}{25}e^{5t} - \frac{t}{5} - \frac{6}{25}$$

Nice thing: get formula for $y(t)$ in terms of $g(t)$

$y'' + 4y = g(t) \qquad g(t)$ any forcing function.

$y(0) = 1 \quad y'(0) = 2$

$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s - 2$

$s^2Y(s) - s - 2 + 4Y(s) = G(s) \qquad G(s) = \mathcal{L}\{g(t)\}$

$(s^2+4)Y(s) = s + 2 + G(s)$

$$Y(s) = \frac{s+2}{s^2+4} + \frac{G(s)}{s^2+4}$$

Inverse $\mathcal{L}^{-1}$: $\quad \dfrac{s+2}{s^2+4} = \dfrac{s}{s^2+4} + \dfrac{2}{s^2+4} \longleftrightarrow \cos 2t + \sin 2t$

$$\frac{G(s)}{s^2+4} = G(s)H(s), \quad H(s) = \frac{1}{s^2+4}$$

$$h(t) = \frac{1}{2}\sin 2t$$

$$\mathcal{L}^{-1}\{H(s)G(s)\} = \int_0^t h(t-\tau)\, g(\tau)\, d\tau$$

$$= \int_0^t \frac{1}{2}\sin[2(t-\tau)]\, g(\tau)\, d\tau$$

Full sol^n: $\quad y(t) = \underbrace{\cos 2t + \sin 2t}_{\text{solns of homog eqn}} + \underbrace{\int_0^t \frac{1}{2}\sin[2(t-\tau)]\, g(\tau)\, d\tau}_{\text{integral transformation of the forcing function.}}$

Fact: $\quad f * g = g * f$

$$\int_0^t f(t-\tau)\, g(\tau)\, d\tau = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$ay'' + by' + cy = g(t) \qquad y(0) = y_0 \quad y'(0) = y_0'$$

$$(as^2 + bs + c)\,Y(s) - (as+b)\,y_0 - a y_0' = G(s)$$

$$Y(s) = \frac{1}{as^2+bs+c}\Big((as+b)\,y_0 + a y_0' + G(s) \Big)$$

$H(s) = \dfrac{1}{as^2+bs+c}$ is called the transfer function of the differential equation.

$h(t) = \mathcal{L}^{-1}\{H(s)\}$ is called the impulse response.

Fact about $h(t)$:

$h(t)$ solves the equation $ay'' + by' + cy = \delta(t)$

$$y_0 = 0 \quad y_0' = 0$$