- Step functions & Discontinuous Functions
- Turning things ON and OFF

$$ay'' + by' + cy = g(t) \leftarrow \text{(external) Forcing function}$$

So far can deal this using $\begin{cases} \text{variation of parameters} \\ \text{undetermined coefficients} \end{cases}$

- Undetermined coefficients need g to have form involving t^n, e^{at}, $\cos at$, $\sin bt$
- Variation of parameters: need to find homogeneous solutions.

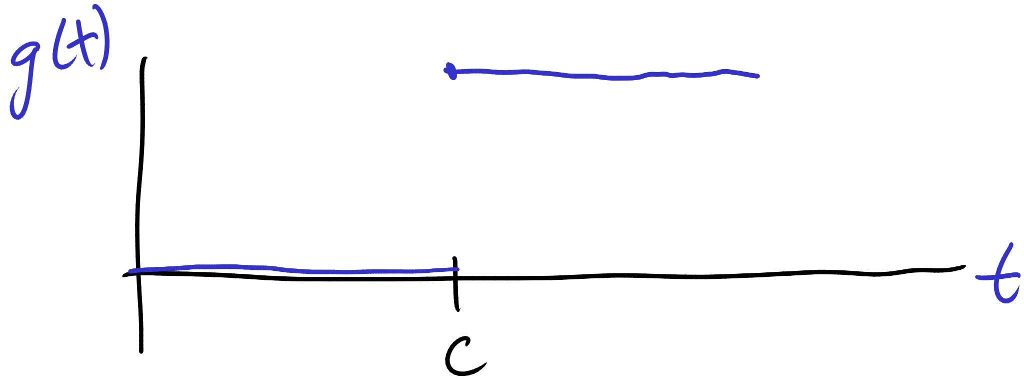
Need to do integrals like $u_1 = \int \frac{-y_2 g(t)}{W(y_1, y_2)} dt$

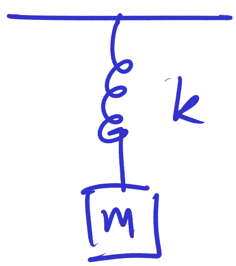
Can be difficult to do this integral

- 3rd method: take Laplace transform of both sides solve for $\mathcal{L}\{y\}$, and take inverse Laplace tranf.

Turning the forcing function on and off

$$ay'' + by' + cy = g(t)$$





damping γ $\quad$ u = displacement

$$mu'' + \gamma u' + ku = 0 \quad \text{when OFF}$$

$$mu'' + \gamma u' + ku = F \quad \text{when ON}$$

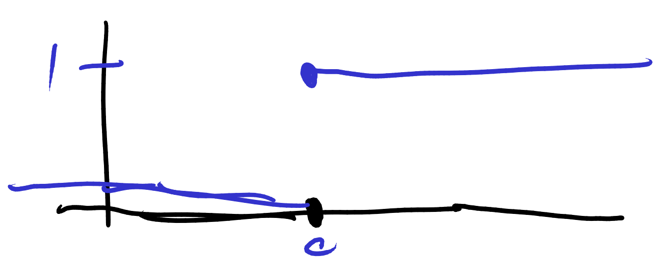
Q: What is the dynamics like when the external force is turned ON?

Define a piecewise continuous function

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

($c \geq 0$ is a number)

$$\boxed{\begin{array}{c} ay'' + by' + cy = u_c(t) \\ \text{this sort of equation} \end{array}}$$



$$\mathcal{L}\{u_c(t)\} = \int_0^\infty u_c(t)\, e^{-st}\, dt$$

$$= \int_0^c u_c(t) e^{-st} dt + \int_c^\infty u_c(t) e^{-st} dt$$

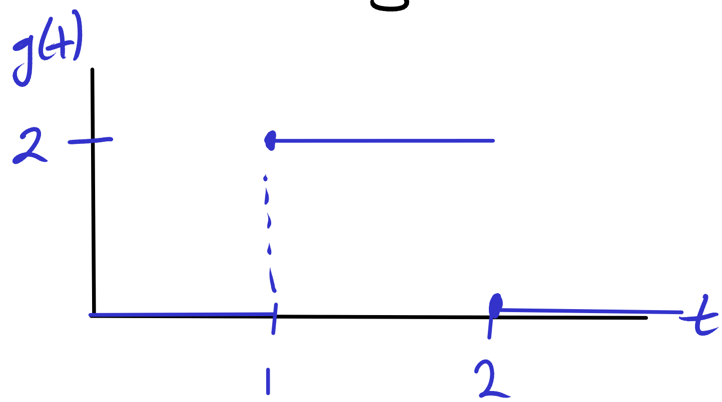
$$= 0 + \int_c^\infty e^{-st} dt = \lim_{B\to\infty} \left[\frac{1}{-s} e^{-st} \right]_{t=c}^{t=B}$$

$$= \lim_{B\to\infty} \frac{1}{-s} \Big(\underbrace{e^{-sB}}_{\to 0 \text{ if } s>0} - e^{-sc} \Big) = \frac{e^{-cs}}{s} \quad (s>0)$$

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s} \quad (s>0)$$

line 12 in the table

Ex:

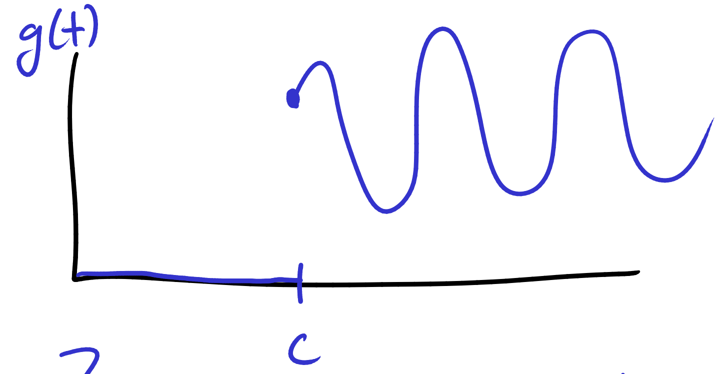


$$g(t) = 2u_1(t) - 2u_2(t)$$

turn on a 2 at t=1

turn on a −2 at t=2

$$\mathcal{L}\{g(t)\} = 2\,\frac{e^{-s}}{s} - 2\,\frac{e^{-2s}}{s} = \frac{2}{s}\left(e^{-s} - e^{-2s}\right)$$

What about a function like 

$$g(t) = \begin{cases} 0 & t<c \\ f(t-c) & t \geq c \end{cases} = u_c(t)\, f(t-c)$$

$u_c(t)$ is called the unit step function (Heaviside step func.)

$$\mathcal{L}\{u_c(t) f(t-c)\} = \int_0^\infty u_c(t) f(t-c) e^{-st}\, dt$$

$$= \int_c^\infty f(t-c)\, e^{-st}\, dt \quad = \int_0^\infty f(u)\, e^{-s(u+c)}\, du$$

$$\boxed{\begin{aligned} u &= t-c \\ t &= u+c \\ dt &= du \end{aligned}} \quad = e^{-sc} \int_0^\infty f(u) e^{-su}\, du$$

$$= e^{-cs} \mathcal{L}\{f(t)\}$$

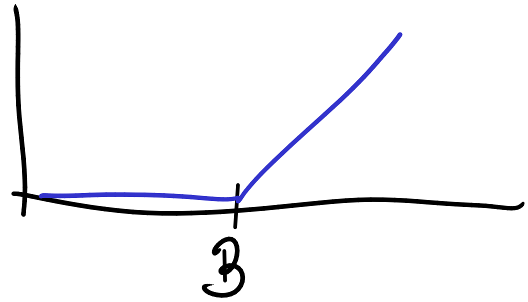
$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} \mathcal{L}\{f(t)\}$$ $\boxed{\text{line 13 in table}}$

Conversely, if $F(s) = \mathcal{L}\{f(t)\}$

$$\mathcal{L}\{u_c(t)\, f(t-c)\} = e^{-cs} F(s)$$

$$u_c(t)\, f(t-c) = \mathcal{L}^{-1}\{e^{-cs} F(s)\}$$

Find $\mathcal{L}\{u_3(t)(t-3)\}$



$= e^{-3s}F(s)$

where $F(s) = \mathcal{L}\{t\}$

$f(t) = t$

$u_3(t)f(t-3) = u_3(t)(t-3)$

$= \frac{1}{s^2}$

$\mathcal{L}\{u_3(t)(t-3)\} = e^{-3s}/s^2$

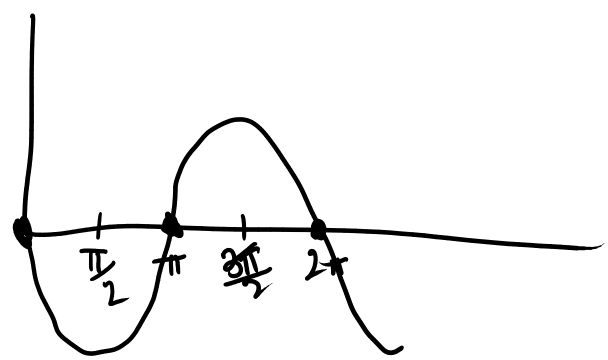
$\mathcal{L}\{u_\pi(t)\sin t\}$

$f(t) = \sin(t+\pi) = -\sin t$

$\mathcal{L}\{u_\pi(t)f(t-\pi)\}$

$f(t-\pi) = \sin t$

$= e^{-\pi s}F(s)$

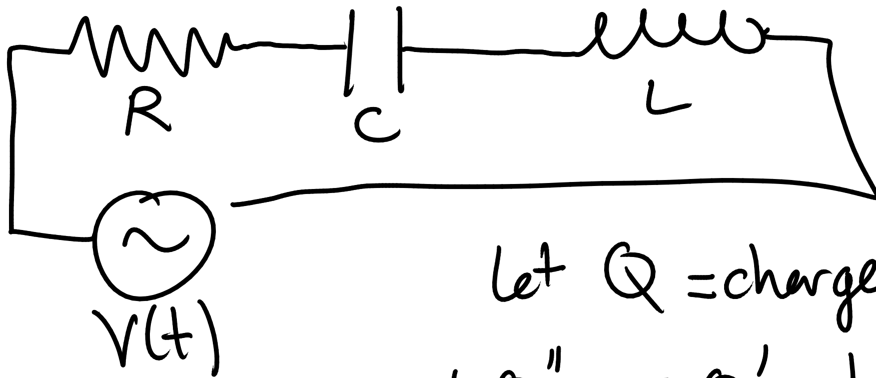


$F(s) = \mathcal{L}\{-\sin t\}$

$= -\frac{1}{s^2+1}$

Answer: $-e^{-\pi s}/(s^2+1)$

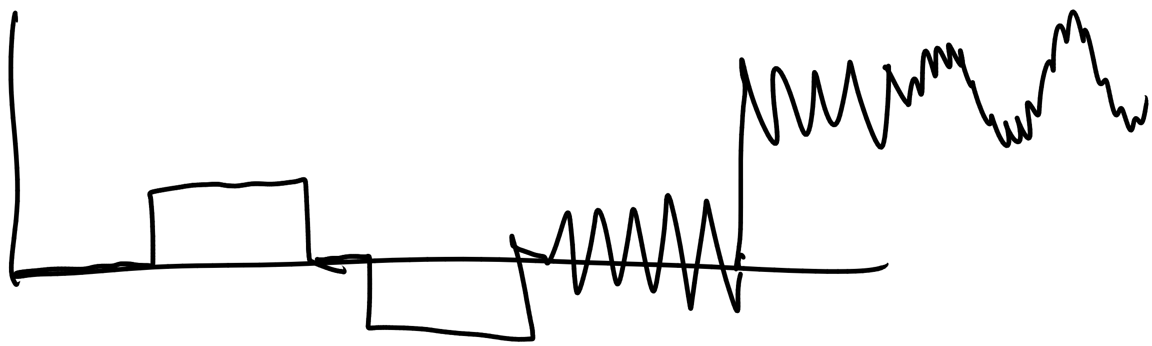
Now we can do lots of situations



Let Q = charge in capacitor:

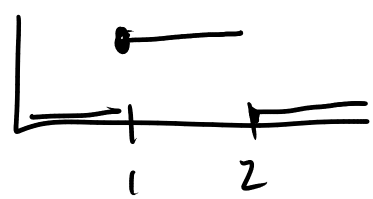
$$LQ'' + RQ' + \frac{1}{C}Q = V(t)$$

Turning



Eg. $y'' + 2y' + 2y = f(t)$ $\quad f(t) = \begin{cases} 0 & t<1,\ t\geq 2 \\ 1 & 1\leq t<2 \end{cases}$

$y(0) = 0 \quad y'(0) = 0$



Introduce $Y(s) = \mathcal{L}\{y(t)\}$

$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY(s)$ $\quad f(t) = u_1(t) - u_2(t)$

$\mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$

$\mathcal{L}\{f(t)\} = (e^{-s} - e^{-2s})/s$

$$s^2Y(s) + 2sY(s) + 2Y(s) = \frac{e^{-s} - e^{-2s}}{s}$$

$$(s^2 + 2s + 2)\,Y(s) = (e^{-s} - e^{-2s})/s$$

$$Y(s) = \frac{e^{-s} - e^{-2s}}{s(s^2+2s+2)}$$ we have the laplace transform of the solution.

Need to take $\mathcal{L}^{-1}$; lets call $H(s) = \frac{1}{s(s^2+2s+2)}$

$$Y(s) = e^{-s}H(s) - e^{-2s}H(s)$$ look at line 13

Suppose $h(t) = \mathcal{L}^{-1}\{H(s)\}$ then

$$\mathcal{L}^{-1}\{e^{-s}H(s)\} = u_1(t)h(t-1)$$

$$\mathcal{L}^{-1}\{e^{-2s}H(s)\} = u_2(t)h(t-2)$$

$$y(t) = u_1(t)h(t-1) - u_2(t)h(t-2)$$

The solution is also defined piecewise, just like the forcing function.

All we need to do now is find $h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\}$

$$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{(s^2+2s+2)}$$

$$1 = A(s^2+2s+2) + (Bs+C)s$$

plug in $s=0$: $1 = A \cdot 2 + 0$ $\quad A = 1/2$

$$1 = As^2 + 2As + 2A + Bs^2 + Cs$$

$As^2 + Bs^2 = 0$ $\qquad$ $2As + Cs = 0$

$A + B = 0 \qquad B = -\frac{1}{2} \qquad 2A + C = 0, C = -1$

$$H(s) = \boxed{\frac{1/2}{s}} + \frac{-\frac{1}{2}s - 1}{s^2+2s+2}$$

$\frac{1}{2}$

Complete the square in s^2+2s+2

$(s+1)^2 + 1 = s^2+2s+2$

Like lines 9 and 10 in table

$$\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$$

Going to match when $a = -1 \quad b = 1$

Need write

$$\frac{-\frac{1}{2}s - 1}{(s+1)^2+1^2} = \alpha\left(\frac{1}{(s+1)^2+1^2}\right) + \beta\left(\frac{s+1}{(s+1)^2+1^2}\right)$$

$\beta s + \beta + \alpha = -\frac{1}{2}s - 1 \qquad \beta = -\frac{1}{2} \quad \alpha = -\frac{1}{2}$

$$\mathcal{L}^{-1}\left\{\frac{-\frac{1}{2}s-1}{(s+1)^2+1^2}\right\} = \alpha\left(e^{-t}\sin t\right) + \beta\left(e^{-t}\cos t\right)$$

$$= -\frac{1}{2}e^{-t}(\cos t + \sin t)$$

$$h(t) = \frac{1}{2} - \frac{1}{2}e^{-t}(\cos t + \sin t)$$

$$y(t) = u_1(t)\,h(t-1) - u_2(t)\,h(t-2)$$