Laplace Transform.

- Definition $\int_0^\infty f(t)\, e^{-st}\, dt = \mathcal{L}\{f(t)\} = F(s)$
- Compute some examples
- Inverting Laplace transform
- Derivative Law for Laplace Transform
  ↳ solve ODE's
- subtle point : when is the Laplace Transform defined?
  ↳ exponential order.

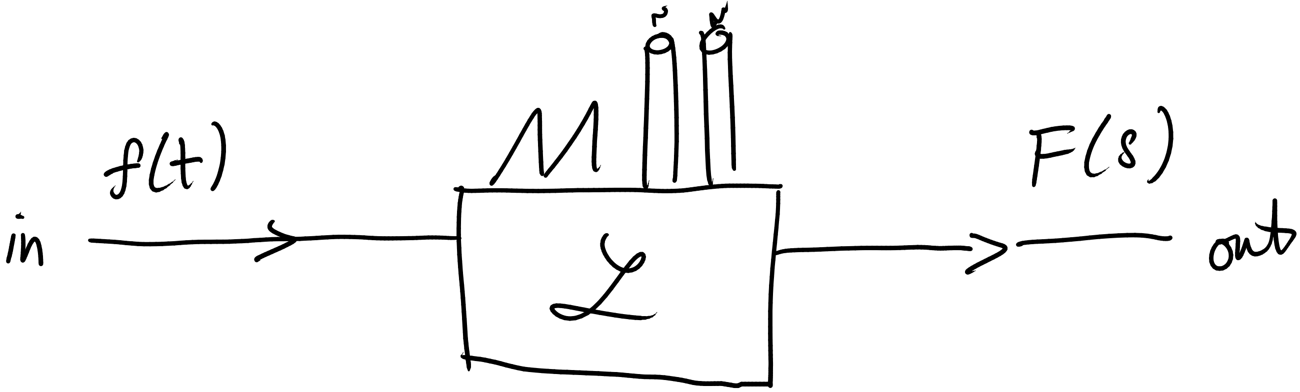
Definition: let $f(t)$ be a function defined for $t > 0$

The Laplace transform of $f(t)$ is

$$\int_0^\infty f(t)\, e^{-st}\, dt = \underbrace{\mathcal{L}\{f(t)\} = F(s)}_{\text{Notations for Laplace Transform.}}$$

s is a variable

t is integrated over, so it doesn't appear in the result.



- The result is a function of s.

If t represents time, then s does not represent time

- s is roughly analogous to "frequency" but it's not that either.

For us s does not have meaning by itself.

- $$\int_0^\infty f(t)e^{-st}\,dt = \lim_{B\to\infty}\int_0^B f(t)e^{-st}\,dt$$

Because it's an improper integral

- Reason why this is useful: Laplace transform converts differential equations into algebraic equations

(Magic)

Examples $\quad \int_0^\infty f(t)\, e^{-st}\, dt$

Ex 1) $\quad f(t) = 1$, a constant function.

$$\mathcal{L}\{1\} = \int_0^\infty 1\, e^{-st}\, dt = \int_0^\infty e^{-st}\, dt$$

$$= \lim_{B\to\infty} \int_0^B e^{-st}\, dt$$

$u = -st$
$du = -s\, dt$
$dt = \frac{1}{-s}\, du$

treat s as constant

$$\int e^{-st}\, dt = \int \frac{1}{-s} e^u\, du = \frac{1}{-s} e^u = \frac{1}{-s} e^{-st}$$

$$= \lim_{B\to\infty} \left[\frac{1}{-s} e^{-st}\right]_{t=0}^{t=B} \overset{\lim}{=} \frac{1}{-s}\left(e^{-sB} - e^{-s0}\right)$$

$$= \lim_{B\to\infty} \frac{1}{-s}\left(e^{-sB} - 1\right) \quad ; \quad e^{-sB} \to 0 \text{ as } B\to\infty$$

$$= \frac{1}{-s}(-1) = \frac{1}{s}$$

ONLY WORKS If $s > 0$
If $s < 0$ integral is divergent.

$$\mathcal{L}\{1\} = \frac{1}{s} \quad \text{for } s > 0$$

What about $s \le 0$? meaningless.

Ex 2: $f(t) = e^{at}$ where a is constant

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{at} e^{-st}\,dt = \int_0^\infty e^{-(s-a)t}\,dt$$

well $\int_0^\infty e^{-st}\,dt = \frac{1}{s}$, $s > 0$

$$\int_0^\infty e^{-(blah)t}\,dt = \frac{1}{blah}, \quad blah > 0$$

$$\int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a}, \quad s-a > 0$$

$s > a$.

Lets say $\mathcal{L}\{f(t)\} = F(s)$ f any function.

What is $\mathcal{L}\{e^{at} f(t)\}$?

$$= \int_0^\infty e^{at} f(t)\, e^{-st}\,dt = \int_0^\infty f(t)\, e^{-(s-a)t}\,dt$$

//

$$F(s-a) = \int_0^\infty f(t)\, e^{-(s-a)t}\,dt$$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

Table of Laplace Transforms (incomplete)

$$f(t) \rightarrow \boxed{} \rightarrow F(s) = \mathcal{L}\{f(t)\}$$

$f(t)$	$F(s)$	
1	$1/s, \quad s>0$	
e^{at}	$1/s-a, \quad s>a$	
$\cos at$	$s/s^2+a^2, \quad s>0$	HW
$\sin at$	$a/s^2+a^2, \quad s>0$	HW
t^n	$n!/s^{n+1}, \quad s>0$	HW

Refer to table on p. 317

Linearity of Laplace transform $\mathcal{L}$

$$\mathcal{L}\{f+g\} = \mathcal{L}\{f\} + \mathcal{L}\{g\}$$

$$\mathcal{L}\{cf\} = c\mathcal{L}\{f\}$$

I.e. $\mathcal{L}$ is a <u>Linear Transformation</u>

function of t $f(t)$ $\longrightarrow$ function of s $F(s) = \mathcal{L}\{f\}$

$$\mathcal{L}\{3\cos 3t + e^{4t}\} = 3\mathcal{L}\{\cos 3t\} + \mathcal{L}\{e^{4t}\}$$

$$= 3\left(\frac{s}{s^2+3^2}\right) + \frac{1}{s-4} \quad (s>4)$$

Linearity is obvious because its an integral

$$\mathcal{L}\{f+g\} = \int_0^\infty (f+g)e^{-st}\,dt = \int_0^\infty f e^{-st}\,dt + \int_0^\infty g e^{-st}\,dt$$

Derivative rule for Laplace transforms.

If f and f' are sufficiently nice (exponential order)
To be defined

then for s sufficiently large

$$\mathcal{L}\{f'(t)\} = s \cdot \mathcal{L}\{f(t)\} - f(0)$$

Relates differentiation to multiplication.

Also works for second derivative:

$$\mathcal{L}\{f''(t)\} = s\,\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s\left(s\mathcal{L}\{f(t)\} - f(0)\right) - f'(0)$$

$$= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

Solve the initial value problem $(y = y(t))$

$$y'' - 2y = 0 \qquad y(0) = 1, \quad y'(0) = 0$$

$$\mathcal{L}\{y'' - 2y\} = \mathcal{L}(0) = 0$$

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y\} = 0$$

$$s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) - 2\mathcal{L}\{y\} = 0$$

$$(s^2 - 2)\,\mathcal{L}\{y\} - s - 0 = 0$$

$$\mathcal{L}\{y\} = \frac{s}{s^2-2}$$

last Q: what is y?

$$y = \mathcal{L}^{-1}\left\{\frac{s}{s^2-2}\right\}$$

Partial Fractions

$$\frac{s}{s^2-2} = \frac{s}{(s+\sqrt{2})(s-\sqrt{2})} = \frac{A}{s+\sqrt{2}} + \frac{B}{s-\sqrt{2}}$$

$$s = A(s-\sqrt{2}) + B(s+\sqrt{2})$$

$$s = (A+B)s + (B-A)\sqrt{2}$$

$A + B = 1$

$A - B = 0$

$$A = B = \frac{1}{2}$$

$$y = \mathcal{L}^{-1}\left\{\frac{s}{s^2-2}\right\} = \mathcal{L}^{-1}\left\{\frac{1/2}{s+\sqrt{2}} + \frac{1/2}{s-\sqrt{2}}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+\sqrt{2}}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-\sqrt{2}}\right\}$$

$$= \frac{1}{2}\, e^{-\sqrt{2}t} + \frac{1}{2}\, e^{\sqrt{2}t}$$

Nonhomogeneous equations

$$y'' - 2y = \sin 2t \qquad y(0)=1,\ y'(0)=0$$

$$\mathcal{L}\{y''-2y\} = \mathcal{L}\{\sin 2t\}$$

$$(s^2-2)\mathcal{L}\{y\} - s = \frac{2}{s^2+4}$$

← algebraic equations in terms of functions of s.

$$\mathcal{L}\{y\} = \underbrace{\frac{s}{s^2-2}}_{\text{did this before}} + \frac{2}{(s^2-2)(s^2+4)}$$

Partial Fractions for $\dfrac{2}{(s^2-2)(s^2+4)}$

$$\frac{2}{(s^2-2)(s^2+4)} = \frac{2}{(s+\sqrt{2})(s-\sqrt{2})(s^2+4)}$$

$$= \frac{A}{s+\sqrt{2}} + \frac{B}{s-\sqrt{2}} + \frac{Cs+D}{s^2+4}$$

$A = \dfrac{-1}{6\sqrt{2}}$ $B = \dfrac{1}{6\sqrt{2}}$ $C = 0$, $D = -\dfrac{1}{3}$

$$Ae^{-\sqrt{2}t} + Be^{\sqrt{2}t} + C\cos 2t + \frac{D}{2}\sin 2t$$

$$y = \left(\frac{1}{2} - \frac{1}{6\sqrt{2}}\right)e^{-\sqrt{2}t} + \left(\frac{1}{2} + \frac{1}{6\sqrt{2}}\right)e^{\sqrt{2}t} - \frac{1}{6}\sin 2t$$