Regular Singular Points

Last time we talked about ordinary points

$$P(x)\, y'' + Q(x) y' + R(x) y = 0$$

$$y'' + p(x) y' + q(x) y = 0 \qquad p(x) = \frac{Q(x)}{P(x)}$$

$$q(x) = \frac{R(x)}{P(x)}$$

if $p(x)$ and $q(x)$ are analytic,
at $x = x_0$.

= have convergent powerseries representations at x_0

Then: there are solutions $y = \sum a_n (x - x_0)^n$
which are convergent power series

E.g. $x^2 y'' + x y' + y = 0$ at $x_0 = 0$

$$y'' + \frac{1}{x} y' + \frac{1}{x^2} y = 0 \quad \text{at } x_0 = 0$$

coefficients go to ∞ as x goes to 0.
$\frac{1}{x}$ and $\frac{1}{x^2}$ are not analytic at 0, so
0 is a singular point of the equation.

But there are solutions for $x > 0$.

Try power series at $x_0 = 0$ $\quad y = \sum_{n=0}^{\infty} a_n x^n$

$$x^2 y'' + x y' + y = 0$$

$$x y' = x \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^n$$

$$x^2 y'' = x^2 \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} n(n-1) a_n x^n$$

$$\sum_{n=0}^{\infty} \left[n(n-1) a_n + n a_n + a_n \right] x^n = 0$$

$$n(n-1) a_n + n a_n + a_n = 0$$

$$\left(n(n-1) + n + 1 \right) a_n = 0$$

$$(n^2 - n + n + 1) a_n = 0$$

$$(n^2+1) a_n = 0$$

$n=0$	$1 a_0 = 0$
$n=1$	$2 a_1 = 0$
$n=2$	$5 a_2 = 0$
$n=3$	$10 a_3 = 0$
$\vdots$	$\vdots$

all a_n's are zero !!!

$y = 0$ We didn't make any progress

There is no power series solution of the form $\sum_{n=0}^{\infty} a_n x^n$. (except 0)

Definition: (Regular singular point)

write the equation as $y'' + p(x)y' + q(x)y = 0$

then x_0 is a regular singular point if it is a singular point, and the two limits

$$\lim_{x \to x_0} (x - x_0) p(x) \quad \text{and} \quad \lim_{x \to x_0} (x - x_0)^2 q(x)$$

exist and are finite.

eg. $y'' + \frac{1}{x} y' + \frac{1}{x^2} y = 0$

$p(x) = \frac{1}{x} \qquad q(x) = \frac{1}{x^2} \qquad x_0 = 0$

$\lim_{x \to 0} x \frac{1}{x}$ and $\lim_{x \to 0} x^2 \frac{1}{x^2}$ exist and are finite

this is regular

Not regular: $y'' + \frac{1}{x^3} y = 0$ at $x = 0$

since $\lim_{x \to 0} x^2 q(x) = \lim_{x \to 0} x^2 \frac{1}{x^3} = \lim_{x \to 0} \frac{1}{x} = \pm \infty$

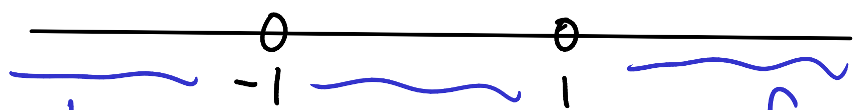
Determine the singular points of

$$(1-x^2)y'' - 2xy' + 10y = 0$$

$$y'' - \frac{2x}{1-x^2}y' + \frac{10}{1-x^2}y = 0$$

singularities when $p(x)$ and $q(x)$ are not defined

i.e. Singular when $1-x^2=0 \iff x = \pm 1$



can solve for $x<-1$ can solve for $-1<x<1$ can solve for $x>1$

Is $x=1$ a regular point?

$$\lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}(x-1)\frac{(-2)x}{1-x^2} = \frac{0}{0}$$

$$\lim_{x\to 1}(x-1)\frac{(-2)x}{(1-x)(1+x)} = \lim_{x\to 1}\cancel{(x-1)}\frac{2x}{\cancel{(x-1)}(x+1)}$$

$$= \lim_{x\to 1}\frac{2x}{x+1} = 1$$

$$\lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}(x-1)^2\frac{10}{1-x^2} = \lim_{x\to 1}(x-1)^2\frac{-10}{(x-1)(x+1)}$$

$$= \lim_{x\to 1}(x-1)\frac{(-10)}{x+1} = 0$$

Both limits finite, so $x=1$ is regular

Basic example of a regular singular point

"Euler Equation"

$$x^2 y'' + \alpha x y' + \beta y = 0 \qquad \left(\begin{array}{c}\text{where } \alpha \,\&\, \beta \text{ are} \\ \text{real constants}\end{array}\right)$$

or $$y'' + \frac{\alpha}{x} y' + \frac{\beta}{x^2} y = 0$$

$x=0$ is a regular singular point

Power series can't solve this.

Let's try $y = x^r$ (consider only $x>0$)

$$y' = r x^{r-1} \qquad y'' = r(r-1) x^{r-2}$$

$$x^2\left(r(r-1)x^{r-2}\right) + \alpha x\left(r x^{r-1}\right) + \beta x^r = 0$$

$$r(r-1)x^r + \alpha r x^r + \beta x^r = 0$$

$$[r(r-1) + \alpha r + \beta] x^r = 0$$

Need r to solve $r(r-1) + \alpha r + \beta = 0$

"Indicial equation"

$$r^2 - r + \alpha r + \beta = 0$$

$$r^2 + (\alpha - 1) r + \beta = 0$$

Note if r is not an integer, then we define

$$x^r = e^{r \ln x}$$

$$\left[x^r = \left(e^{\ln x}\right)^r = e^{r \ln x}\right]$$

The cases are parallel to constant coefficient second order equations.

<u>Real and distinct roots:</u> $x^2y''+2xy'-2y=0$

Try x^r: $x^2(r)(r-1)x^{r-2}+2xrx^{r-1}-2x^r=0$

$$[r(r-1)+2r-2]\,x^r=0$$

Need $r(r-1)+2r-2=0$

$$r^2+r-2=0$$

solutions $r_1=1$ $r_2=-2$

this means that x^1 and x^{-2} are solutions for $x>0$

$$W(x,x^{-2})=\begin{vmatrix} x & x^{-2} \\ 1 & -2x^{-3}\end{vmatrix}=-3x^{-2}\neq 0 \text{ for } x>0$$

so $y_1=x$ and $y_2=x^{-2}$ are a fundamental set

general solution $y=C_1x+C_2x^{-2}$

In general, if r_1 and r_2 are distinct real roots of $r(r-1)+\alpha r+\beta=0$

then the general solution to $x^2y''+\alpha xy'+\beta y=0$ is

$$y=c_1x^{r_1}+c_2x^{r_2} \quad (x>0)$$

$$y=c_1|x|^{r_1}+c_2|x|^{r_2} \quad (x<0 \text{ as well})$$

Possible graphs

Real repeated roots: $x^2y''+3xy'+y=0$

try x^r $\quad [r(r-1)+3r+1]x^r=0$

indicial equation is $\quad r(r-1)+3r+1=0$

$$r^2-r+3r+1=0$$

$$r^2+2r+1=0 \quad \text{so } \boxed{r_1=r_2=-1}$$

$$\overset{\shortparallel}{(r+1)^2}$$

$y_1=x^{-1}$ $\qquad y_2=x^{-1}\ln x$ is the other solution

(could use reduction of order to derive this)

for general repeated root, solution is

$$y=c_1x^{r_1}+c_2x^{r_1}\ln x \qquad (x>0)$$

$$y=c_1|x|^{r_1}+c_2|x|^{r_1}\ln|x| \qquad (x\neq 0)$$

Complex conjugate roots $\quad x^2y'' + xy' + y = 0$

try x^r $\quad [r(r-1) + r + 1]x^r = 0$

$$r(r-1) + r + 1 = 0$$

$$r^2 + 1 = 0 \qquad r_1 = i \quad r_2 = -i$$

Complex solutions $\quad x^i$ and x^{-i}

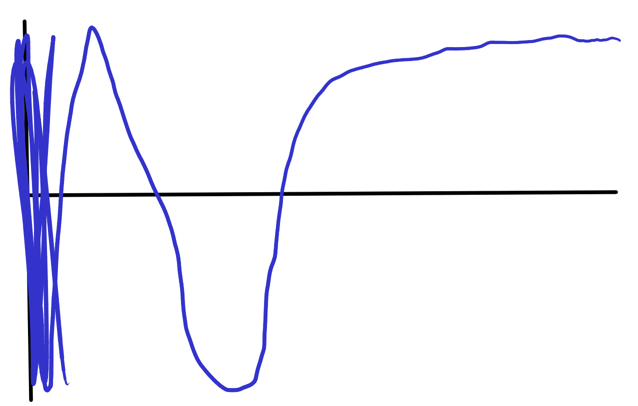
$$x^i = e^{i \ln x} = \cos(\ln x) + i\sin(\ln x)$$

$$x^{-i} = e^{-i\ln x} = \cos(-\ln x) + i\sin(-\ln x)$$

$$= \cos(\ln x) - i\sin(\ln x)$$

Fundamental set of real solutions:

$$y_1 = \cos(\ln x) \qquad y_2 = \sin(\ln x)$$



general complex roots $\quad r_1 = \lambda + i\mu \quad r_2 = \lambda - i\mu$

complex solution $\quad x^{(\lambda+i\mu)} = e^{(\lambda+i\mu)\ln x} = e^{\lambda \ln x} e^{i\mu \ln x}$

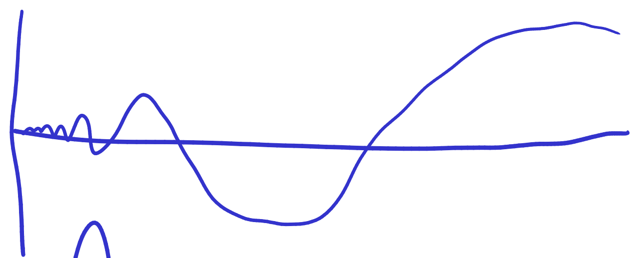
$$= x^{\lambda}\left(\cos(\mu \ln x) + i \sin(\mu \ln x)\right)$$

$$x^{(\lambda - i\mu)} = x^{\lambda}\left(\cos(\mu \ln x) - i \sin(\mu \ln x)\right)$$

General real solution

$$y = x^{\lambda}\left(c_1 \cos(\mu \ln x) + c_2 \sin(\mu \ln x)\right)$$

$\lambda > 0$



$\lambda < 0$