Exam Stats

	Raw	Curved
Average	70.7	80.5
1 Q	61	75
Median	70	80
3 Q	80	86.7

curved $= 100 - \left(\frac{2}{3}\right)(100 - \text{Raw})$

Last time: Power series, radius of convergence, differentiating power series

Solved $\quad y' - y = 0 \qquad y'' + y = 0$

$$y = a_0 e^x \qquad y = a_0 \cos x + a_1 \sin x$$

Today: some non constant coefficient equations. (solutions are not exponential)

Example $\quad y'' - xy = 0 \quad$ find power series solutions centered at $x_0 = 0$.

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} a_n x^n \qquad \text{(Airy's equation)}$$

Recall $\quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$

$$xy = x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1} x^n$$

$$y'' - xy = \sum_{\substack{n=0 \\ \uparrow \\ \text{starts at } 0}}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{\substack{n=1 \\ \uparrow \\ \text{starts at } 1.}}^{\infty} a_{n-1}x^n$$

$$= \underbrace{2\cdot 1\cdot a_2}_{n=0 \text{ term}} + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n$$

$$= 2a_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - a_{n-1}\right]x^n$$

Want this to be zero: $2a_2 = 0 \to a_2 = 0$

and $(n+2)(n+1)a_{n+2} - a_{n-1} = 0$ for $n = 1, 2, \ldots$

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)} \quad \text{recurrence relation.}$$

always for 2nd order linear

a_0 can be anything $\to a_3 = \frac{a_0}{2\cdot 3} \to a_6 = \frac{a_3}{5\cdot 6} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6}$

a_1 can be anything $\to a_4 = \frac{a_1}{3\cdot 4} \to a_7 = \frac{a_4}{6\cdot 7} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7}$

$a_2 = 0 \longrightarrow a_5 = \frac{a_2}{4\cdot 5} = 0 \to a_8 = 0 \to a_{11} = 0 \to \cdots$

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continue $a_9 = \underbrace{\frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}}_{\text{kind of like a factorial.}}$ $\quad a_{10} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}$

1st chain $n = 3k$ $2^n n! = 2\cdot 4\cdot 6\cdot 8 \cdots 2n$

$$a_{3k} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdots(3k-1)(3k)}$$ $\leftarrow$ all proportional to a_0

2nd chain $n = 3k+1$

$$a_{3k+1} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdots(3k)(3k+1)}$$ $\leftarrow$ all proportional to a_1

3rd chain $n = 3k+2$

$$a_{3k+2} = 0.$$

General solution:

$$y = \overbrace{a_0\left[1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \cdots + \frac{x^{3k}}{2\cdot 3\cdots(3k-1)(3k)} + \cdots\right]}^{y_1}$$

$$\underbrace{+\, a_1\left[x + \frac{x^4}{3\cdot 4} + \frac{x^7}{3\cdot 4\cdot 6\cdot 7} + \cdots + \frac{x^{3k+1}}{3\cdot 4\cdots(3k)(3k+1)} + \cdots\right]}_{y_2}$$

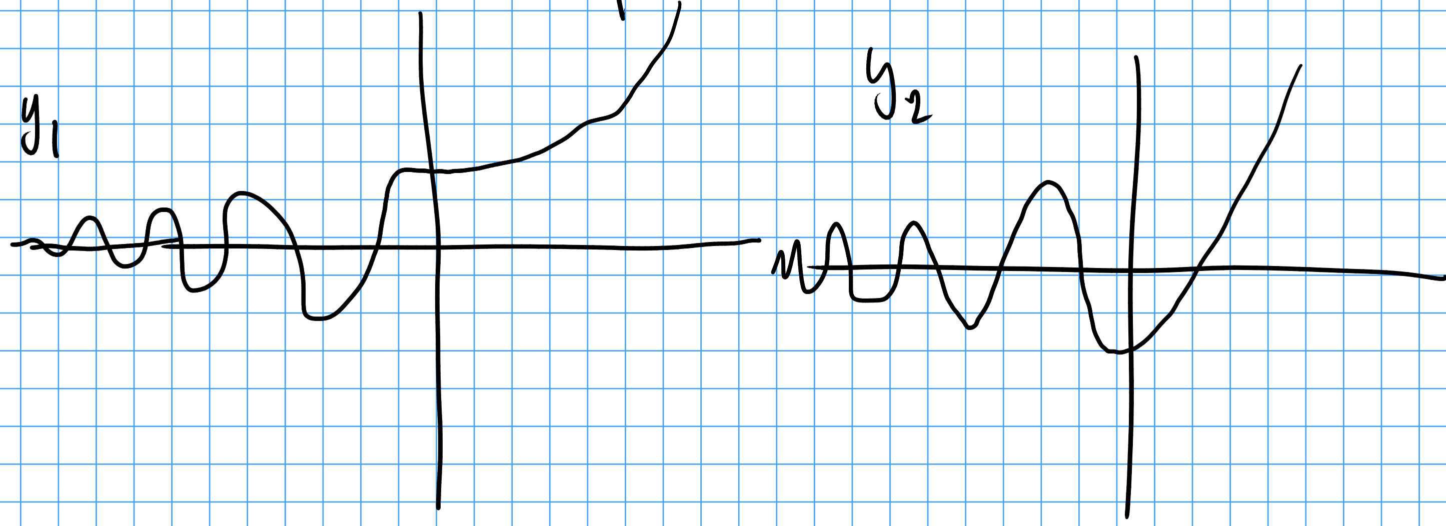
$$y = a_0 y_1 + a_1 y_2$$

y_1 and y_2 are a fundamental set of solutions.

Compute Wronskian at $x=0$. $y_1(0) = 1$ $y_1'(0) = 0$

$y_2(0) = 0$ $y_2'(0) = 1$

$$W(y_1,y_2)(x=0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1\cdot 1 - 0\cdot 0 = 1 \neq 0$$



Theoretical aspects of power series solutions.

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0 \quad \text{General form}$$

$$y'' + p(x)\,y' + q(x)\,y = 0 \quad \text{Standard form.}$$

$$p(x) = Q(x)/P(x) \quad q(x) = R(x)/P(x)$$

Fact $\quad y = \sum a_n (x-x_0)^n \Rightarrow n!\,a_n = y^{(n)}(x_0)$

Plug in $x = x_0$ in the ODE

$$y''(x_0) + p(x_0)\,y'(x_0) + q(x_0)\,y(x_0) = 0$$

$$2a_2 + p(x_0)\,a_1 + q(x_0)\,a_0 = 0$$

Solve for a_2 given $a_1 = y'(x_0)$ $\quad a_0 = y(x_0)$

To find a_3, differentiate the ODE

$$\frac{d}{dx}\left(y'' + p(x)y' + q(x)y = 0 \right)$$

$$y''' + p'(x)y' + p(x)y'' + q'(x)y + q(x)y' = 0$$

$$y'''(x_0) + p'(x_0)y'(x_0) + p(x_0)y''(x_0) + q'(x_0)y(x_0) + q(x_0)y'(x_0) = 0$$

$$3!\,a_3 + p'(x_0)\,a_1 + p(x_0)(2a_2) + q'(x_0)\,a_0 + q(x_0)\,a_1 = 0$$

Knowing $a_0, a_1, a_2, p(x_0), q(x_0), p'(x_0), q'(x_0)$,
can get a_3

Can get as many a_n's as we want, but we need to keep computing derivatives of p and q.

(ex $y'' + \sin(x)\,y = 0$)

In order to work, need $p(x)$ and $q(x)$ to be infinitely differentiable at $x = x_0$

To get convergent power series solution, actually need p and q to be <u>analytic at x_0</u>

<u>Definition</u> a function $f(x)$ is analytic at x_0 if it has a power series representation

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with a positive radius of convergence.

Therem $y'' + p(x)y' + q(x)y = 0$

if p and q are analytic at x_0

then there are convergent power series solutions centered at x_0

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad y = a_0 y_1 + a_1 y_2$$

where $y_1(x_0) = 1 \quad y_1'(x_0) = 0$

$y_2(x_0) = 0 \quad y_2'(x_0) = 1$

And the radius of convergence of the solution is at least as large as the minimum of the radii of convergence of p and q.

E.g. $y'' - xy = 0$ A polynomial is analytic, with radius of convergence $= \infty$

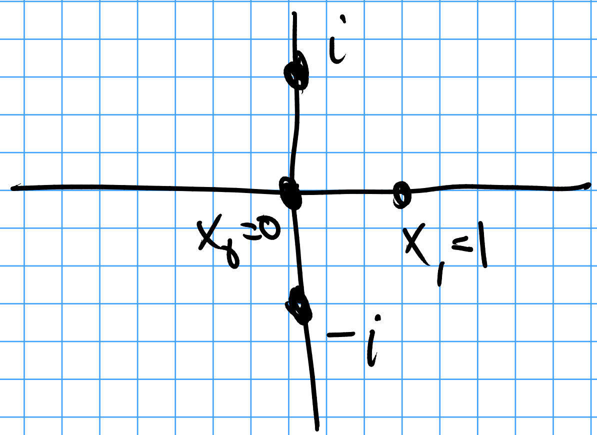
$$f(x) = \frac{A(x)}{B(x)}$$ ratio of polynomials in reduced form (no common factor)

zero of $B(x)$ is called a pole of $f(x)$

radius of convergence at x_0 is the distance from x_0 to the nearest pole, which may be complex

Ex $y'' + \frac{1}{1+x^2} y = 0 \qquad q(x) = \frac{1}{1+x^2}$

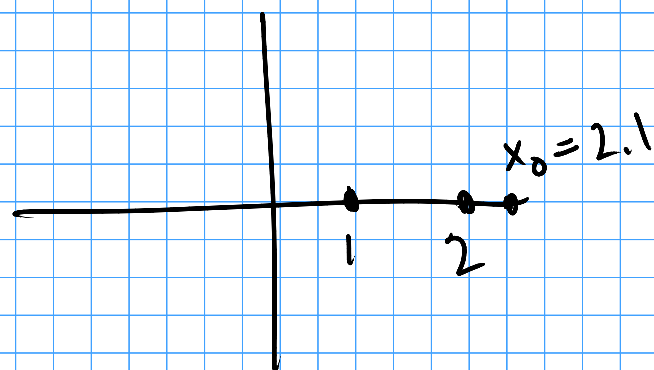
poles = zeros of $1+x^2$ = i or $-i$



radius of conv of $q(x)$ at $x_0 = 0$ is $\boxed{1}$

radius of conv at $x_1 = 1$ is $\sqrt{2}$

$$\frac{1}{(x-2)(x-1)}$$



radius of conv $= .1$

Ex $(2x^2-1)y'' + xy' + y = 0 \qquad x_0 = 0$

How big is the radius of convergence of the solution guaranteed to be?

$$y'' + \frac{x}{2x^2-1}y' + \frac{1}{2x^2-1}y = 0$$

$2x^2-1=0 \qquad x = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$

radius of convergence of $p(x)$ and $q(x)$ at $x_0 = \frac{\sqrt{2}}{2}$

so solution has radius of convergence $\geq \frac{\sqrt{2}}{2}$