Ch5 Power series solutions

So far mainly equations w/ constant coefficients

$$ay'' + by' + cy = g(t)$$

$$y'' + p(t)y' + q(t)y = 0$$

$$R(x)\frac{dy^2}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Example : $\frac{d^2y}{dx^2} - xy = 0$ Airy's Diff. Eqn.

Study second order linear nonconstant coeff.

Basic Idea : look for solutions of the form

$$y = a_0 + a_1x + a_2x^2 + \cdots$$

ending index

coefficients

center of series

$$= \sum_{n=0}^{\infty} \boxed{a_n}(x - \boxed{x_0})^n$$

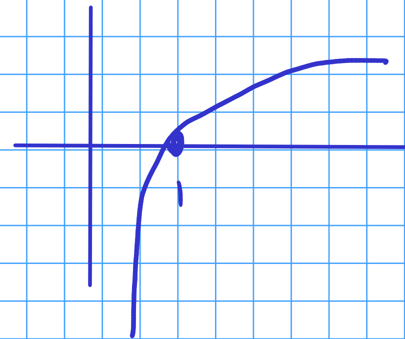
n = index of summation

starting index

Ex $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1} + \frac{x^2}{1\cdot 2} + \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^4}{1\cdot 2\cdot 3 4} + \cdots$

$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$$\underbrace{\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}}$$



formula valid for

$0 < x \leq 2$

Convergence $\quad \sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges at x

if $\lim_{N\to\infty} \sum_{n=0}^{N} a_n (x-x_0)^n$ exists.

Absolute convergence: $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ converges absolutely

if $\sum_{n=0}^{\infty} |a_n| |x-x_0|^n$ converges

Use ratio test: $\sum_{n=0}^{\infty} b_n$

look at $L = \lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|}$

if $L < 1$ then series converges absolutely

if $L > 1$ then series diverges

if $L = 1$ inconclusive

Apply to $\sum_{n=0}^{\infty} a_n (x-x_0)^n$

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|\,|x-x_0|^{n+1}}{|a_n|\,|x-x_0|^n} = \left(\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}\right)|x-x_0|$$

To get convergence, need $L<1$

$$\left(\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}\right)|x-x_0| < 1$$

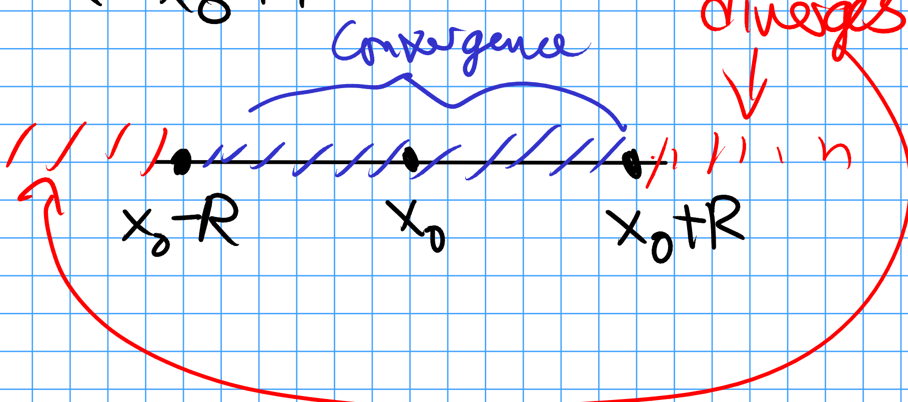
solve for x

$$|x-x_0| < \left(\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}\right)^{-1} =: R = \text{Radius of convergence}$$

$$-R < x-x_0 < R$$

$$x_0 - R < x < x_0 + R$$

graph this set:



<u>Example</u>: $\sum_{n=0}^{\infty} 2^n x^n$ center $= x_0 = 0$

$$L = \lim_{n\to\infty} \frac{2^{n+1}|x|^{n+1}}{2^n |x|^n} = \lim_{n\to\infty} 2|x| = 2|x|$$

$L < 1$ if and only if $2|x| < 1$

$|x| < \frac{1}{2} = R =$ radius of convergence

$-\frac{1}{2} < x < \frac{1}{2}$

$\ln x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$ center $= x_0 = 1$

$$\lim_{n\to\infty} \frac{\frac{|x-1|^{n+1}}{n+1}}{\frac{|x-1|^n}{n}} = \left(\lim_{n\to\infty} \frac{n}{n+1}\right) |x-1| = |x-1|$$

converges when $|x-1| < 1$

$-1 < x-1 < 1$

$0 < x < 2$

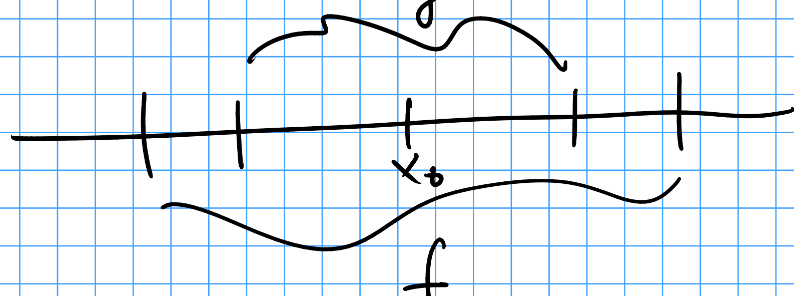
$\sum_{n=0}^{\infty} n!\,x^n$ radius of convergence $= 0$

Within the Interval of convergence, power series work like polynomials

$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \qquad g(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n$

suppose both converge on intervals containing

$|x-x_0| < \rho_g$



$$f(x)+g(x) = \sum_{n=0}^{\infty} (a_n+b_n)(x-x_0)^n$$

valid where both f and g converge

multiply (infinite FOIL)

$$f(x)g(x) = \left[\sum_{n=0}^{\infty} a_n(x-x_0)^n\right]\left[\sum_{n=0}^{\infty} b_n(x-x_0)^n\right]$$

$$= \sum_{n=0}^{\infty} c_n(x-x_0)^n$$

$$c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0$$

$$= \sum_{k=0}^{n} a_kb_{n-k}$$

Can divide power series, but interval of convergence may be smaller.

Differentiation: $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$

Take Derivative term by term (valid within interval of converge)

$$f'(x) = \sum_{n=0}^{\infty} a_n \cdot n \cdot (x-x_0)^{n-1}$$

$$f''(x) = \sum_{n=0}^{\infty} a_n \cdot n \cdot (n-1)(x-x_0)^{n-2}$$

Taylor coefficients: $a_n = \dfrac{f^{(n)}(x_0)}{n!}$

Uniqueness If $f=g$ then $a_n = b_n$ for all n

If $f=0$ then $a_n = 0$ for all n

Shift of index of summation

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \sum_{j=0}^{\infty} \frac{2^j x^j}{j!}$$

index of summation is a dummy variable.

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \sum_{k=1}^{\infty} \frac{2^{k-1} x^{k-1}}{(k-1)!}$$

$$\boxed{\begin{array}{l} k = n+1 \\ n = k-1 \end{array}}$$

Use this trick $\displaystyle f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$

$$f'(x) = \sum_{n=0}^{\infty} n\, a_n (x - x_0)^{n-1} \underset{\substack{k=n-1 \\ n=k+1}}{=\!=} \sum_{k=-1}^{\infty} (k+1)\, a_{k+1} (x - x_0)^k$$

$$\underset{k=n}{=\!=} \sum_{n=-1}^{\infty} (n+1)\, a_{n+1} (x - x_0)^n \quad \left(\begin{array}{c}\text{different} \\ n\end{array}\right)$$

$$= \underbrace{(\cancel{-1+1})}_{0}\, a_{-1+1} (x - x_0)^{-1} + \sum_{n=0}^{\infty} (n+1)\, a_{n+1} (x - x_0)^n$$

Final answer $\displaystyle f'(x) = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} (x - x_0)^n$

$$f''(x) = \sum_{n=0}^{\infty} (n+1)(n+2)\, a_{n+2} (x - x_0)^n$$

Example! $\quad y' - y = 0 \qquad \boxed{y = ce^x}$

Step! suppose solution is a power series centered at x_0 (deferring the question of convergence)

Let's take $x_0 = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$0 \stackrel{?}{=} y' - y = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} \underbrace{\left[(n+1) a_{n+1} - a_n\right]}_{\text{need this } = 0 \text{ for every } n} x^n$$

We get $(n+1) a_{n+1} - a_n = 0$ for every $n = 0, 1, \dots$

$$a_{n+1} = \frac{a_n}{(n+1)} \quad \text{for every } n = 0, 1, 2 \dots$$

$a_0 = \text{something} \qquad a_1 = \frac{a_0}{1} \qquad a_2 = \frac{a_1}{2} = \frac{a_0}{1 \cdot 2}$

$a_3 = \frac{a_2}{3} = \frac{a_0}{1 \cdot 2 \cdot 3} \qquad a_4 = \frac{a_3}{4} = \frac{a_0}{1 \cdot 2 \cdot 3 \cdot 4}$

(Recurrence relation) $\qquad a_n = \frac{a_0}{1 \cdot 2 \cdot 3 \cdots n} = \frac{a_0}{n!}$

Solution $y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n$

$$= \boxed{a_0} \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x$$

arbitrary constant in the general solution.

Should check for convergence at this point.

Example 2 $\quad y'' + y = 0 \qquad y = c_1 \cos x + c_2 \sin x$

suppose $\quad y = \sum_{n=0}^{\infty} a_n x^n \qquad y'' = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$

$$y'' + y = \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n \right] x^n = 0$$

Need $\quad (n+2)(n+1) a_{n+2} + a_n = 0 \quad$ for every n

Recurrence Relation $\quad a_{n+2} = \frac{-a_n}{(n+1)(n+2)}$

$a_0, \quad a_2 = \frac{-a_0}{1 \cdot 2}, \quad a_4 = \frac{-a_2}{3 \cdot 4} = \frac{a_0}{1 \cdot 2 \cdot 3 \cdot 4}, \quad a_6 = \frac{-a_0}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$

$a_1, \quad a_3 = \frac{-a_1}{2 \cdot 3}, \quad a_5 = \frac{-a_3}{4 \cdot 5} = \frac{a_1}{2 \cdot 3 \cdot 4 \cdot 5}, \quad a_7 = \frac{-a_1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}$

a_{even} : $a_{2n} = \frac{(-1)^n}{(2n)!} a_0$

a_{odd} : $a_{2n+1} = \frac{(-1)^n}{(2n+1)!} a_1$

$$y = a_0 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}_{\cos x} + a_1 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}_{\sin x}$$

$$y = a_0 \cos x + a_1 \sin x$$