Higher order Linear ODE's

- Exam 1 next Tuesday on Chapters 1-3 (not Today's lecture)
- Special office hours Monday 12-2 (instead of Tuesday 3:30-5:00)
- No quiz next Wednesday.

Higher order linear equation

STANDARD FORM

$$\begin{cases} \dfrac{d^n y}{dt^n} + p_1(t)\dfrac{d^{n-1} y}{dt^{n-1}} + p_2(t)\dfrac{d^{n-2} y}{dt^{n-1}} + \cdots + p_{n-1}(t)\dfrac{dy}{dt} + p_n(t)y \\ \quad = g(t) \end{cases}$$

$$\frac{d^n y}{dt^n} = y^{(n)} = D^n y \qquad D = \frac{d}{dt}$$

raising operator to nth power means repeat operator n times

- can sometimes think of D as an algebraic symbol.

$$D^n y + p_1(t) D^{n-1} y + \cdots + p_{n-1}(t) D y + p_n(t) y = g(t)$$

$$\left[D^n + p_1(t) D^{n-1} + \cdots + p_{n-1}(t) D + p_n(t)\right] y = g(t)$$

What does the initial value problem look like?

There are n initial conditions

$$y(t_0) = y_0 \quad y'(t_0) = y_0' , \quad y''(t_0) = y_0'' , \; y'''(t_0) = y_0''',$$

$$\cdots \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

Basic fact: when coefficients $p_1(t), p_2(t), \ldots, p_n(t), g(t)$ are continuous, the initial value problem always has a unique solution.

Homogeneous $\left[D^n + p_1(t) D^{n-1} + \cdots + p_n(t)\right] y = 0$

Eg. $D^n y = f(t)$ $\quad y = \underbrace{\int\int\int}_{n \text{ times}} f(t)$

get n constants of integration

general solution has n constants.

In general, the homogeneous eqn has n "independent" solutions $y_1, \ldots, y_n$ and the general solution is

$$y = C_1 y_1 + C_2 y_2 + \cdots + C_{n-1} y_{n-1} + C_n y_n$$

Supersimple example $D^n y = 0$

$$y^{(n)} = 0$$

$y_1 = 1$, $y_2 = t$, $y_3 = t^2$, $\cdots$, $y_n = t^{n-1}$

$$1 \xrightarrow{D} 0 \qquad t \xrightarrow{D} 1 \xrightarrow{D} 0 \qquad t^2 \xrightarrow{D} 2t \xrightarrow{D} 2 \xrightarrow{D} 0 \qquad t^{n-1} \xrightarrow{D^n} 0$$

general solution $y = c_1 \cdot 1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}$

that is any degree $n-1$ polynomial.

Consider $y''' + y' = 0$

$$(D^3 + D)y = 0$$

$$D(D^2+1)y = 0$$

$$\left\{ \begin{array}{l} (D^2+1)y = 0 \\ y'' + y = 0 \end{array} \right]$$

check $y_1 = 2\cos t$ is a solutions

$$y_1 \xrightarrow{D^2+1} 0 \xrightarrow{D} 0$$

$y_2 = 6\sin t$ is a solution

$y_3 = 3\cos t + 5\sin t$ is a solution

Three solutions, but NOT GOOD ENOUGH

Eg. Try to solve $y(0) = 1$ $y'(0) = 0$ $y''(0) = 0$

can't do it with any combination of these solutions in fact $y_4 = 1$ is also a solution

and $u_1 = \cos t$ $\quad u_2 = \sin t$ $\quad u_3 = 1$ is a "complete" (fundamental) set of solutions

↳ can solve any IVP using these

Two criteria for having a fundamental set:

Wronskian criterion

$y_b^{(a)}$ = a th derivative of the b th solution.

$$c_1 y_1 + \cdots + c_n y_n = y_0$$
$$c_1 y_1' + \cdots + c_n y_n' = y_0'$$
$$\vdots$$
$$c_1 y_1^{(n-1)} + \cdots + c_n y_n^{(n-1)} = y_0^{(n-1)}$$

$n \times n$ system of linear equations.

→ linear algebra says we need the determinant

$$W(y_1, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \neq 0$$

For 3x3
$$\begin{vmatrix} A & B & C \\ D & E & F \\ G & H & I \end{vmatrix} = A \begin{vmatrix} E & F \\ H & I \end{vmatrix} - B \begin{vmatrix} D & F \\ G & I \end{vmatrix} + C \begin{vmatrix} D & E \\ G & H \end{vmatrix}$$

For $n \geq 4$ use Gaussian elimination or a computer.

<u>Linear independence</u>:

functions $y_1, \dots, y_n$ are Not linearly independent (linearly dependent)

if there are constants $k_1, k_2, \dots, k_n$ <u>not all zero</u>, such that

$$k_1 y_1 + \cdots + k_n y_n = 0$$

linearly independent $\Leftrightarrow$ not not linearly independent

$$\left(\frac{3}{2}\right)(2\cos t) + \left(\frac{5}{6}\right)(6\sin t) + (-1)(3\cos t + 5\sin t) = 0$$

not linearly independent

$$k_1 \cos t + k_2 \sin t + k_3 \cdot 1 = 0$$

linearly independent, to show this, need to show that $k_1 = 0, k_2 = 0, k_3 = 0$ is the only possibility.

Plug in $t=0$ $\quad k_1 \cdot 1 + k_2 \cdot 0 + k_3 \cdot 1 = 0 \Rightarrow k_1 + k_3 = 0$

Plug in $t = \frac{\pi}{2}$ $\quad k_1 \cdot 0 + k_2 \cdot 1 + k_3 \cdot 1 = 0 \Rightarrow k_2 + k_3 = 0$

Plug in $t = \pi$ $\quad k_1(-1) + k_2 \cdot 0 + k_3 \cdot 1 = 0 \Rightarrow -k_1 + k_3 = 0$

$\Rightarrow k_1 = 0, k_2 = 0, k_3 = 0.$

$y_1, \dots, y_n$ is a fundamental set

if and only if

$W(y_1, \dots, y_n) \not\equiv 0$

if and only if

$y_1, \dots, y_n$ are linearly independent.

Nonhomogeneous equation

$$D^n y + p_1(t) D^{n-1} y + \dots + p_n(t) y = g(t)$$

$y_1, \dots, y_n$ fundamental set of homogeneous solutions

Y any particular nonhomogeneous solution

General nonhomogeneous solutions

$$y = c_1 y_1 + \dots + c_n y_n + Y$$

Main example: Linear constant coefficient equations

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0$$

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_{n-1} D + a_n) y = 0$$

Try e^{rt} $\quad De^{rt} = re^{rt}$ $\quad D^n e^{rt} = r^n e^{rt}$

$D^2 e^{rt} = r^2 e^{rt}$

($De^{rt} = re^{rt}$ says that e^{rt} is an eigenvector for D)

$$0 \stackrel{?}{=} (a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n) e^{rt}$$

$$= \underbrace{(a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n)}_{\text{need this expression to be zero}} e^{rt}$$

Characteristic equation

$$a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0$$

degree n polynomial equation.

$n > 2$ it's not easy or even impossible to solve for r using $\sqrt[n]{\ }$

But it is still true that any polynomial (with real coefficients) factors into a product of linear polynomials and irreducible quadratics.

$$(r - r_1)(r - r_2) \cdots (r - r_k)$$

$$\times (r^2 + b_1 r + c_1)(r^2 + b_2 r + c_2)$$

$$b_1^2 - 4c_1 < 0 \qquad b_2^2 - 4c_2 < 0$$

If r_1 is a real root $y_1 = e^{r_1 t}$ is a solution if $r^2+b_1r+c_1$ is a factor in the characteristic equation.

$$r = \frac{-b_1 \pm \sqrt{b_1^2-4c_1}}{2} = \lambda \pm \mu i$$

complex solutions $e^{(\lambda+\mu i)t}$, $e^{(\lambda-\mu i)t}$

real solutions $e^{\lambda t}\cos(\mu t)$, $e^{\lambda t}\sin(\mu t)$

if r_1 is real repeated n_1 times, then get

$$e^{r_1 t},\ te^{r_1 t},\ t^2 e^{r_1 t},\ \cdots,\ t^{n_1-1}e^{r_1 t}$$

$n=12$

$$(D-1)(D-3)(D+50)^4(D^2+1)^2(D^2-2D+2)y=0$$

$r=1$	$r=3$	$r=-50$	$r=\pm i$	$r=1\pm i$
e^t	e^{3t}	e^{-50t}	$\cos t$	$e^t\cos t$
		te^{-50t}	$\sin t$	$e^t\sin t$
		t^2e^{-50t}	$t\cos t$	
		t^3e^{-50t}	$t\sin t$	