Variation of Parameters

Nonhomogeneous $L[y] = y'' + p(t)y' + q(t)y = g(t)$

Homogeneous version $L[y] = y'' + p(t)y' + q(t)y = 0$

general solution to nonhomog. is

$$y = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

$Y(t)$ any particular nonhomog. solution

y_1, y_2 fundamental set of homog. solutions

Variation of Parameters is general method for finding particular solutions of non-homog equations.

To start assume $Y = u_1(t)y_1(t) + u_2(t)y_2(t)$

$$Y' = \underbrace{u_1 Y_1' + u_2 Y_2'}_{\text{keep these}} + \underbrace{u_1' Y_1 + u_2' Y_2}_{\text{require these to be } 0.}$$

Get one condition $\boxed{u_1' Y_1 + u_2' Y_2 = 0}$

$$Y'' = (u_1 Y_1' + u_2 Y_2')'$$

$$= u_1 Y_1'' + u_2 Y_2'' + u_1' Y_1' + u_2' Y_2'$$

Plug into $L[Y] = Y'' + pY' + qY \stackrel{?}{=} g$

$$u_1 Y_1'' + u_2 Y_2'' + u_1' Y_1' + u_2' Y_2'$$

$$p(u_1 Y_1' + u_2 Y_2') + q(u_1 Y_1 + u_2 Y_2) \stackrel{?}{=} g$$

Collect u_1 and u_2 terms

$$u_1(\underbrace{y_1''+py_1'+qy_1}) + u_2(\underbrace{y_2''+py_2'+qy_2})$$
$$+\ u_1'y_1' + u_2'y_2' \overset{?}{=} g$$

these terms are 0 because y_1 and y_2 solve homog. equation.

Second condition is
$$u_1'y_1' + u_2'y_2' = g$$

Get equations for u_1' and u_2'

$$\left\{\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= g \end{aligned}\right\} \text{ can solve this for } u_1' \text{ and } u_2'$$

Solution $\quad u_1' = \dfrac{-y_2\, g}{W(y_1,y_2)}$

$$u_2' = \frac{y_1\, g}{W(y_1,y_2)}$$

Recall
$$W(y_1,y_2) = y_1y_2' - y_2y_1'$$
$$= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Basic linear algebra

$$ax+by=0 \qquad\qquad x = \frac{-bg}{ad-bc}$$
$$cx+dy=g \qquad\qquad y = \frac{ag}{ad-bc}$$

last step integrate $\quad u_1 = \displaystyle\int \frac{-y_2\, g}{W}\, dt$

$$u_2 = \int \frac{y_1\, g}{W}\, dt$$

Plug into $\quad y = u_1y_1 + u_2y_2$

<u>Example:</u> $y'' - 5y' + 6y = 2e^t$

Solve homogeneous problem $\quad r^2 - 5r + 6 = 0$

$(r-2)(r-3) = 0$

$y_1 = e^{2t} \qquad y_2 = e^{3t}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = (3-2)e^{(3+2)t} = e^{5t}$$

Try to find nonhomog solution $y = u_1 e^{2t} + u_2 e^{3t}$

$$y' = u_1(2e^{2t}) + u_2(3e^{3t}) + \underbrace{u_1' e^{2t} + u_2' e^{3t}}_{\text{require } = 0}$$

$$y'' = u_1(4e^{2t}) + u_2(9e^{3t}) + u_1'(2e^{2t}) + u_2'(3e^{3t})$$

Plug into $\quad y'' - 5y' + 6y = 2e^t$

$$u_1'(2e^{2t}) + u_2'(3e^{3t}) = 2e^t$$

$$\begin{cases} u_1' e^{2t} + u_2' e^{3t} = 0 \\ u_1'(2e^{2t}) + u_2'(3e^{3t}) = 2e^t = g \end{cases}$$

$$u_1' = \frac{-y_2 g}{W} = -\frac{e^{3t}(2e^t)}{e^{5t}} = -2e^{-t}$$

$$u_2' = \frac{y_1 g}{W} = \frac{e^{2t}\, 2e^t}{e^{5t}} = 2e^{-2t}$$

$$u_1 = \int -2e^{-t}\,dt = 2e^{-t}$$

$$u_2 = \int 2e^{-2t}\,dt = -e^{-2t}$$

$$y = (2e^{-t})\, e^{2t} + (-e^{-2t})(e^{3t})$$
$$= 2e^{t} + (-1)e^{t} = e^{t}$$

Example $y'' + y = \tan t$

Homogeneous solution: $y_1 = \cos t \quad y_2 = \sin t$

$r^2 + 1 = 0 \quad r = \pm i \quad e^{it}, e^{-it}$

$$W(y_1, y_2) = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

$$y = u_1 \cos t + u_2 \sin t$$

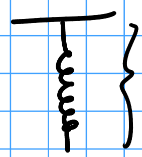
$\hookrightarrow$
$$u_1' = \frac{-y_2 g}{W} = -\sin t \cdot \tan t = -\frac{\sin^2 t}{\cos t}$$

$$u_2' = \frac{y_1 g}{W} = \cos t \tan t = \sin t$$

$$u_1 = \int -\frac{\sin^2 t}{\cos t}\, dt = \ln(\sec t + \tan t) - \sin t$$

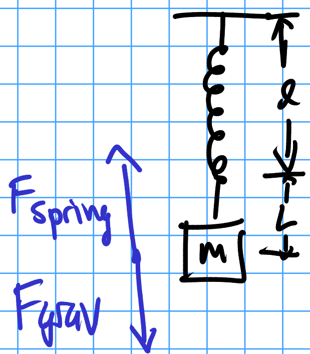
$$u_2 = \int \sin t \, dt = -\cos t$$

Mass on a spring



unstrected length $= \ell$

Put a mass m on it



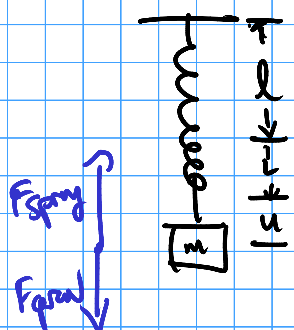
Find equilibrium position.

$$F_{grav} + F_{spring} = 0$$

$$mg - kL = 0 \qquad L = \frac{mg}{k}$$

$k =$ Hooke's law spring constant

Let u be a perterbation from equilibrium



$$F_{spring} = -k(L+u)$$

Down is positive

$$F_{grav} = mg$$

$u =$ position $\quad u' =$ velocity $\quad u'' =$ acceleration

Equation of motion $\quad mu'' = F_{total} = mg - k(L+u)$

$$mu'' = -ku$$

Free undamped $\rightarrow$ $mu'' + ku = 0$

Damping is a force which dissipates the energy

$$F_{damp} = -\gamma u'$$

$$mu'' = -\gamma u' - ku$$

Free damped $\rightarrow$ $mu'' + \gamma u' + ku = 0$

External Force $F(t)$

$$mu'' + \gamma u' + ku = F(t)$$ Forced damped

Compare $ay'' + by' + cy = g(t)$

Free undamped $mu'' + ku = 0$

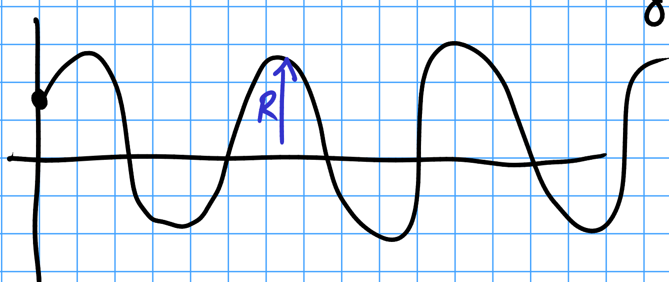
characteristic equation $mr^2 + k = 0$

$$r^2 = -\frac{k}{m}, \quad r = \pm\sqrt{\frac{k}{m}}\, i$$

Term: $\omega_0 = \sqrt{\frac{k}{m}}$ is called natural (angular) frequency

$T = \frac{2\pi}{\omega_0}$ natural period

$$u = A\cos\omega_0 t + B\sin\omega_0 t$$
$$= R\cos(\omega_0 t - \delta) \qquad R = \sqrt{A^2+B^2}$$
$$\delta = \tan^{-1}(B/A)$$



use $u(t_0) = u_0$
$u'(t_0) = u_0'$
to determine constants

Free damped $mu'' + \gamma u' + ku = 0$

$$mr^2 + \gamma r + k = 0$$

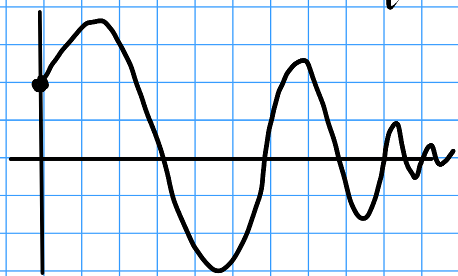
$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

Behavior depends on sign of $\gamma^2 - 4km$

Case 1: $\gamma^2 - 4km < 0$

$$r = \frac{-\gamma}{2m} \pm i \frac{\sqrt{4km - \gamma^2}}{2m} = \mu$$

$$y = A e^{-\gamma t/2m} \cos(\mu t) + B e^{-\gamma t/2m} \sin(\mu t)$$

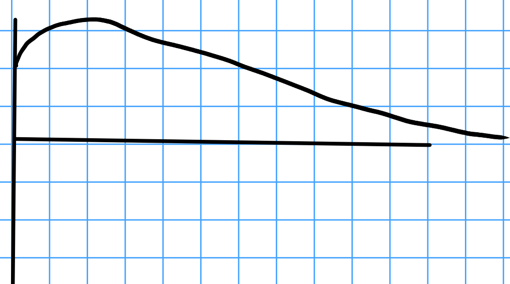


Case 2: $\gamma^2 - 4km = 0$

$$r = -\frac{\gamma}{2m} \quad \text{(repeated root)}$$

$$y = (A + Bt) e^{-\gamma t/2m}$$

(Critically Damped)

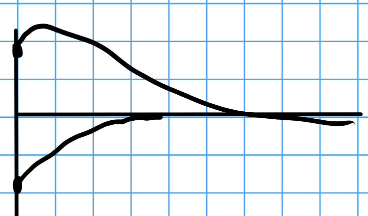


Case 3: $\gamma^2 - 4km > 0$

$$r_1 = \frac{-\gamma}{2m} + \frac{\sqrt{\gamma^2 - 4km}}{2m} \qquad r_2 = \frac{-\gamma}{2m} - \frac{\sqrt{\gamma^2 - 4km}}{2m}$$

Both r_1 and r_2 are negative

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$ overdamped



Forcing $\quad mu'' + \gamma u' + ku = R\cos(\alpha t)$

get sinusoidal particular solution
with frequency α