M 427 K Lecture 1

§ 1.1 & 1.2

Differential Equations.

$y' = f(t, y)$ for an unknown function $y(t)$

$\overset{\shortparallel}{\frac{dy}{dt}}$ $\Rightarrow$ find $y(t)$.

eg. ① $y' = t \Rightarrow y(t) = \frac{1}{2}t^2 + C$

$$\int y' \, dt = \int t \, dt$$
$$y = \frac{1}{2}t^2 + C$$

② $y' = y$ $\qquad \frac{d}{dt} e^t = e^t$

so $y = Ce^t$ for a constant C.

Solve $\frac{dy}{dt} = y$

Separate variables $\rightarrow dy = y\,dt \Rightarrow \frac{dy}{y} = dt$

$\int \frac{dy}{y} = \int dt \rightarrow \ln|y| = t + K$

$$|y| = e^{t+K} = e^t e^K$$

$$y = \pm e^K e^t$$

Call $C = \pm e^K$ (Note $e^K > 0$)

$$y = Ce^t$$

Notice we can let C be negative, positive or 0.

Other examples

$y' = t^2 - y$ $\quad$ $y' = y^2 - t$

easier $\quad$ harder

DE's model physical, biological, economical systems where quantities depend on time.

Eg Newton 2nd law $F = ma$

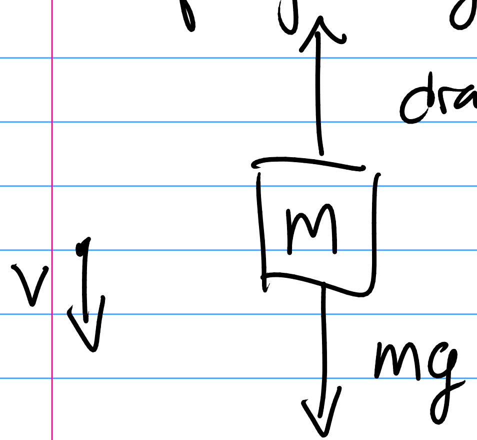
$a = \dfrac{dv}{dt}$ $\quad$ $v = \dfrac{dy}{dt}$

y - position
v - velocity
a - acceleration
m - mass
F - force

$$a = \frac{d^2y}{dt} = y''$$

$$y'' = \frac{F}{m}(t, y)$$

Consider an object falling under influence of gravity and drag.



drag $= \gamma v$ $\qquad \gamma$ = gamma = coeff of drag

$$m\frac{dv}{dt} = mg - \gamma v$$

$$v' = g - \frac{\gamma}{m} v$$

Another example predator-prey dynamics

$p(t)$ = population of mice

$$\underbrace{\frac{dp}{dt} = rp}_{\text{natural growth}} - e \quad \leftarrow \text{amount eaten by the owl.}$$

r = const of proportionality

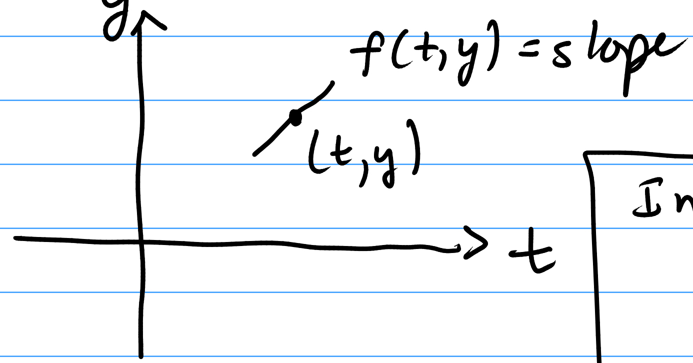
$$p' = rp - e$$

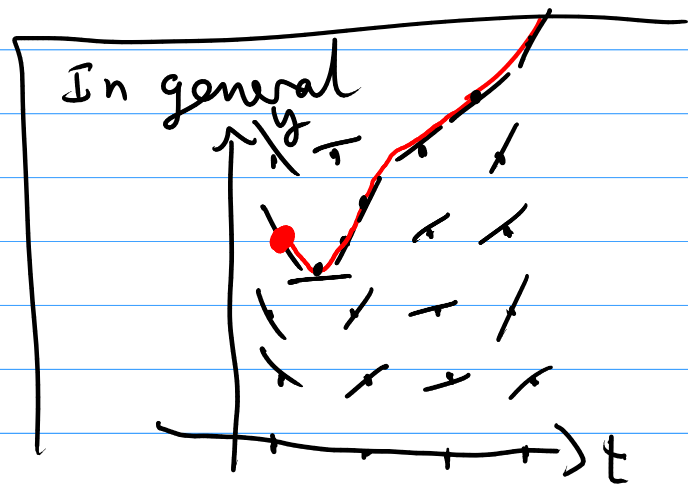
Both are examples of $\quad y' = ay - b$

Geometric View of $y' = f(t,y)$

Direction Field

Def The direction field consists of, at each point (t,y) in the plane, a small line segment with slope equal to $f(t,y)$

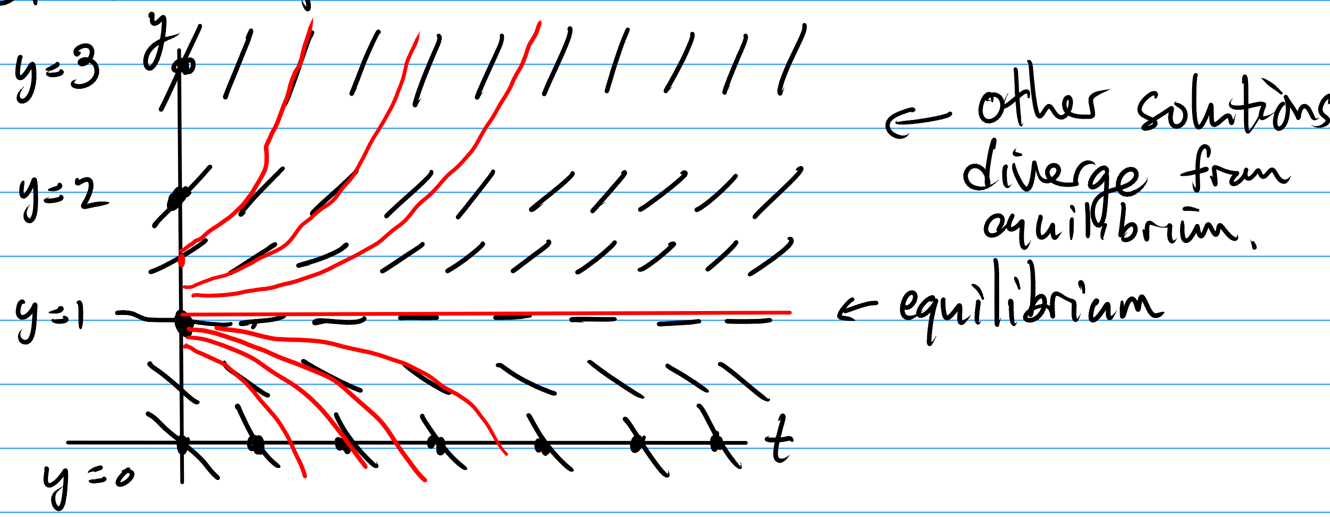




Def A solution curve (integral curve) of the direction field is a path in the (t,y)-plane which is, at every point, tangent to the line segment of the direction field at that point.

Try $y' = f(t,y) = y - 1$

Direction field



There are infinitely many solutions
→ corresponds to the undetermined constant in the general.

<u>Thm</u> $y(t)$ solves DE $\iff$ graph of $y(t)$ is a solution curve for the direction field

$y'(t) = f(t, y(t))$

$\Downarrow$

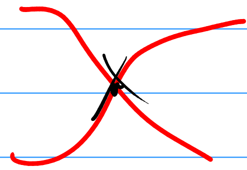
at each point on graph $(t, y(t))$

$y'(t)$ $=$ The slope of the graph equals slope of line segment equals $f(t, y(t))$

by def of derivative

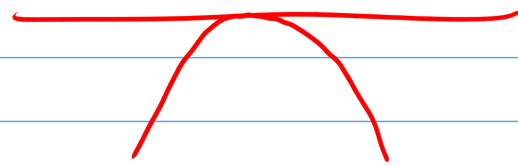
General facts:

① Solution curves can never cross



can't happen

② Solution curves cannot be tangent to one another



can't happen.

Provided that $f(t,y)$ and $\frac{\partial f}{\partial y}(t,y)$ are continuous.

In fact
There is exactly one solution through each point (t,y)

What is the solution to $y' = f(t,y)$

such that $y(0) = y_0$ ← given

$y(0) = y_0$ initial condition

This problem is called the initial value problem.

Falling w/ drag

$$\frac{dv}{dt} = -\frac{\gamma}{m} v + g = \left(-\frac{\gamma}{m}\right)\left(v - \frac{mg}{\gamma}\right)$$

$$\int \frac{dv}{\left(v - \frac{mg}{\gamma}\right)} = \int -\frac{\gamma}{m}\, dt$$

$$\ln\left| v - \frac{mg}{\gamma} \right| = -\frac{\gamma}{m} t + K$$

$$v - \frac{mg}{\gamma} = \underbrace{te^{K}}_{C} e^{-\gamma t/m}$$

$$v(t) = \frac{mg}{\gamma} + C e^{-\gamma t/m}$$

TERMINAL VELOCITY

equilibrium

other solutions converge to equilibrium