M 427K PRACTICE FOR FINAL EXAM

These problems are representative of the problems that will be on the final exam. This list of problems is *longer* than the actual exam will be. All of these problems are similar to problems that were assigned for homework. A copy of the table of Laplace transforms on p. 317 of the textbook will be provided during the exam. In addition, you are permitted one two-sided sheet of notes (US Letter size paper: 8.5"x11"). The notes must be handwritten, and no photocopying is allowed. No other aids (books, calculators) are permitted.

1. First order equations.

   (a) Solve the initial value problem for $y(t)$, assuming $t > 0$.

   $$ty' + 2y = t^2 - t + 1, \qquad y(1) = \frac{1}{2} \tag{1}$$

   Hint: integrating factor

   (b) Find the general solution.

   $$\frac{dy}{dx} = \frac{x^2}{y(1+x^3)} \tag{2}$$

   (c) Determine (without solving the problem) an interval in which the solution of the initial value problem is guaranteed to exist.

   $$t(t-4)y' + y = 0, \qquad y(2) = 1 \tag{3}$$

   (d) Consider the autonomous equation

   $$\frac{dy}{dt} = \frac{-2\arctan y}{1+y^2} \tag{4}$$

   Sketch the graph of the right-hand side, determine the critical (equilibrium) points, and determine whether each is stable, unstable, or semistable. Sketch the graphs of several solutions in the ty-plane.

   (e) Determine whether the equation is exact. If it is, find the solution.

   $$(2xy^2 + 2y) + (2x^2y + 2x)y' = 0 \tag{5}$$

   (f) If y solves the equation $y' = 2y - 1$, and $y(0) = 1$, determine $y(0.2)$ using two steps of Euler's method with a step size of $h = 0.1$.

2. Second order equations.

   (a) Solve the initial value problem.

   $$y'' + 3y' = 0, \qquad y(0) = -2, \qquad y'(0) = 3 \tag{6}$$

   (b) Find the Wronskian of the functions

   $$y_1(t) = e^{-2t}, \qquad y_2(t) = te^{-2t} \tag{7}$$

   (c) Use Euler's formula for complex exponentials to write e^{2-3i} in the form $a + bi$.

(d) Find the general solution.

$$y'' - 2y' + 6y = 0 \tag{8}$$

(e) Find the general solution.

$$4y'' - 4y' - 3y = 0 \tag{9}$$

(f) Find the general solution. Hint: undetermined coefficients.

$$y'' - 2y' - 3y = -3te^{-t} \tag{10}$$

(g) Find a particular solution. Hint: variation of parameters.

$$y'' - 2y' + y = \frac{e^t}{1+t^2} \tag{11}$$

3. Higher order equations.

(a) Determine whether these 4 functions are linearly independent:

$$f_1(t) = 2t - 3, \quad f_2(t) = t^3 + 1, \quad f_3(t) = 2t^2 - t, \quad f_4(t) = t^2 + t + 1 \tag{12}$$

(b) Find the general solution of the eigth-order equation

$$y^{(8)} + 8y^{(4)} + 16y = 0 \tag{13}$$

4. Power series.

(a) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \tag{14}$$

(b) Re-index this series so that the general term involves x^n:

$$x\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n \tag{15}$$

(c) Seek a power series solution of the following equation at the point $x_0 = 0$:

$$(1+x^2)y'' - 4xy' + 6y = 0 \tag{16}$$

Find the recurrence relation, and determine the first four terms of the solution that begins with $a_0 = 1$, $a_1 = 1$.

(d) Find the general solution for $x > 0$ of the differential equation:

$$2x^2y'' - 4xy' + 6y = 0 \tag{17}$$

Note: this equation is an Euler equation with a singular point at $x_0 = 0$.

5. Laplace transform. (A copy of the table on p. 317 of the textbook will be provided during the exam.)

(a) Here are some functions of s. Find their inverse Laplace transforms. The answer may have discontinuities or delta functions in it.

$$F(s) = \frac{2s-3}{s^2+2s+10} \tag{18}$$

$$F(s) = \frac{(s-2)e^{-s}}{s^2-4s+3} \tag{19}$$

$$F(s) = \frac{1}{(s+1)^2(s^2+4)} \tag{20}$$

(b) For each of following equations, solve for $Y(s) = \mathcal{L}\{y(t)\}$.

$$y'' - 4y' + 4y = 0, \quad y(0) = 10, \quad y'(0) = 5 \tag{21}$$

$$y'' - 2y' + 2y = \sin(100t), \quad y(0) = -1, \quad y'(0) = 1 \tag{22}$$

$$y'' + 4y = u_3(t) - u_4(t), \quad y(0) = 0, \quad y'(0) = 0 \tag{23}$$

$$y'' + 2y' + 2y = \delta(t-\pi) + \delta(t-2\pi) + \delta(t-3\pi), \quad y(0) = 0, \quad y'(0) = 0 \tag{24}$$

6. First order systems.

(a) In each of these two cases, find the eigenvalues and eigenvectors of the matrix $\mathbf{A}$, and find the general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \tag{25}$$

$$\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \tag{26}$$

7. Fourier series and partial differential equations

(a) Either solve the boundary value problem or show that it has no solution.

$$y'' + 2y = 0, \qquad y'(0) = 1, \qquad y'(\pi) = 0 \tag{27}$$

(b) Find the eigenvalues and eigenfunctions of the boundary value problem. Assume all the eigenvalues are real.

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \qquad y(\pi) = 0 \tag{28}$$

(c) Find the fundamental period of the function defined by

$$f(x) = \begin{cases} (-1)^n, & 2n-1 \le x < 2n \\ 1, & 2n \le x < 2n+1 \end{cases} \tag{29}$$

In this definition, n ranges over all integers.

(d) Find the Fourier series of the periodic function with period $2L$ defined on the interval $-L \le x < L$ by

$$f(x) = \begin{cases} 1, & -L \le x < 0 \\ 0, & 0 \le x < L \end{cases} \tag{30}$$

(e) Is $f(x) = \sec x$ even, odd, or neither?

(f) Find the sine series (with period 6π) of the function

$$f(x) = \begin{cases} 0, & 0 < x < \pi \\ 1, & \pi < x < 2\pi \\ 2, & 2\pi < x < 3\pi \end{cases} \tag{31}$$

(g) Find the solution of the heat equation

$$\frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2} \tag{32}$$

on the interval $0 < x < \pi$, with boundary conditions $u(0,t) = 0$, $u(\pi,t) = 0$, and initial temperature distribution

$$u(x,0) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin nx \tag{33}$$

(h) Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 100\frac{\partial^2 u}{\partial x^2} \tag{34}$$

on the interval $0 < x < L$, with the boundary conditions $u(0,t) = 0$, $u(L,t) = 0$, and the initial conditions

$$u(x,0) = \sin\frac{5\pi x}{L}, \qquad \frac{\partial u}{\partial t}(x,0) = 0 \tag{35}$$