M 427K PRACTICE FOR FINAL EXAM

These problems are representative of the problems that will be on the final exam. This list of problems is *longer* than the actual exam will be. All of these problems are similar to problems that were assigned for homework. A copy of the table of Laplace transforms on p. 317 of the textbook will be provided during the exam. In addition, you are permitted one two-sided sheet of notes (US Letter size paper: 8.5"x11"). The notes must be handwritten, and no photocopying is allowed. No other aids (books, calculators) are permitted.

1. First order equations.

(a) Solve the initial value problem for $y(t)$, assuming $t > 0$.

$$ty' + 2y = t^2 - t + 1, \qquad y(1) = \frac{1}{2} \tag{1}$$

Hint: integrating factor

$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$ $\qquad \mu = e^{\int \frac{2}{t}dt} = e^{2\ln t} = t^2$

mult by μ: $\quad t^2y' + 2ty = t^3 - t^2 + t$

$(t^2y)' = t^3 - t^2 + t$

$t^2y = \int(t^3 - t^2 + t)\,dt = \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + C$

$y = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + Ct^{-2}$. $\qquad y(1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C = \frac{1}{2}$

$\Rightarrow C = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ $\quad \Rightarrow y = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12}t^{-2}$

(b) Find the general solution.

Separable:

$$\frac{dy}{dx} = \frac{x^2}{y(1+x^3)} \tag{2}$$

$\int y\,dy = \int \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| = \frac{1}{3}\ln|1+x^3|$

$\begin{pmatrix} u = 1+x^3 \\ du = 3x^2dx \end{pmatrix}$

$\frac{1}{2}y^2 = \frac{1}{3}\ln|1+x^3| + C$

$y = \pm\left(\frac{2}{3}\ln|1+x^3| + \underbrace{2C}_{\text{just a constant}}\right)^{1/2}$

(c) Determine (without solving the problem) an interval in which the solution of the initial value problem is guaranteed to exist.

$$t(t-4)y' + y = 0, \qquad y(2) = 1 \tag{3}$$

$$y' + \frac{1}{t(t-4)}y = 0$$

The coefficient $\frac{1}{t(t-4)}$ is continuous except at $t=0$, $t=4$

Since our initial value is at $t=2$, the solution will exist forward to $t=4$ and back to $t=0$.

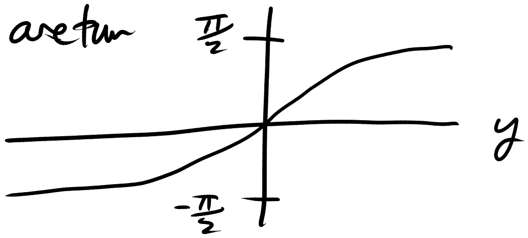
So guaranteed to exist for $0 < t < 4$

(d) Consider the autonomous equation

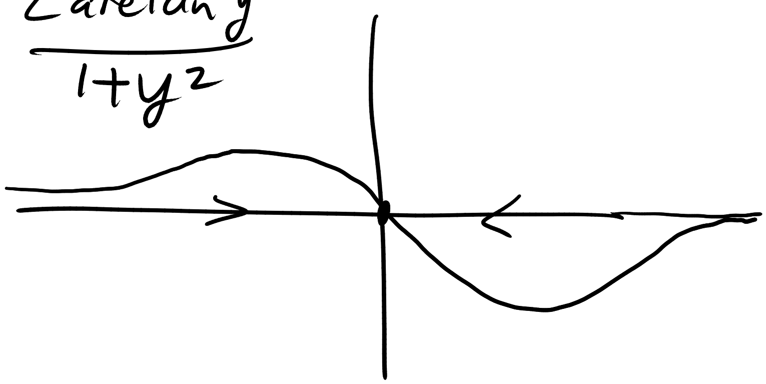
$$\frac{dy}{dt} = \frac{-2\arctan y}{1+y^2} \tag{4}$$

Sketch the graph of the right-hand side, determine the critical (equilibrium) points, and determine whether each is stable, unstable, or semistable. Sketch the graphs of several solutions in the ty-plane.

arctan

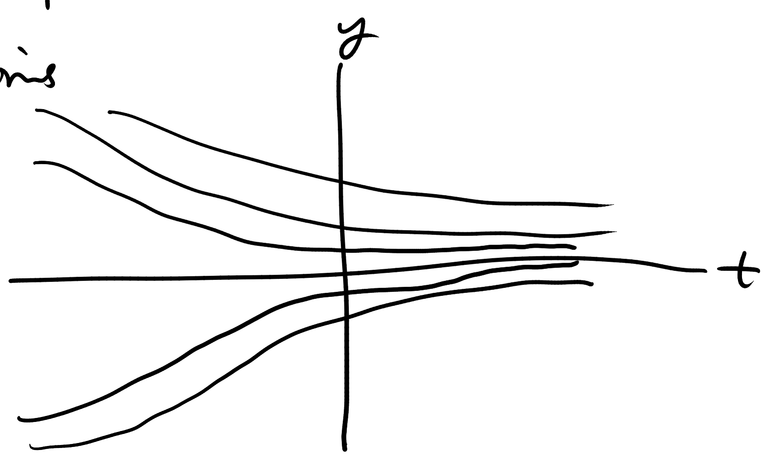


$$-\frac{2\arctan y}{1+y^2}$$



critical point at $y=0$
stable

Sketch of solutions



(e) Determine whether the equation is exact. If it is, find the solution.

$$\underbrace{(2xy^2 + 2y)}_{M} + \underbrace{(2x^2y + 2x)}_{N}y' = 0 \qquad (5)$$

$$\frac{\partial M}{\partial y} = 4xy+2 \qquad \frac{\partial N}{\partial x} = 4xy+2$$

these are equal, so it is exact.

Want $F(x,y)$ so that $M = \frac{\partial F}{\partial x}$, $N = \frac{\partial F}{\partial y}$

$$F = \int M\,dx = \int (2xy^2 + 2y)\,dx = x^2y^2 + 2xy + h(y)$$

$$F = \int N\,dy = \int (2x^2y + 2x)\,dy = x^2y^2 + 2xy + g(x)$$

Can take $h(y) = 0$ and $g(x) = 0$.

Solutions given implicitly by $\boxed{x^2y^2 + 2xy = C}$

(f) If y solves the equation $y' = 2y - 1$, and $y(0) = 1$, determine $y(0.2)$ using two steps of Euler's method with a step size of $h = 0.1$.

$$y(0.1) \approx y(0) + y'(0)(0.1)$$
$$= 1 + (2\cdot 1 - 1)(0.1) = 1 + 0.1 = 1.1$$

$$y(0.2) \approx y(0.1) + y'(0.1)\cdot(0.1)$$
$$= (1.1) + (2(1.1) - 1)\cdot(0.1)$$
$$= (1.1) + (1.2)(0.1)$$
$$= 1.1 + 0.12 = 1.22$$

$$\boxed{y(0.2) \approx 1.22}$$

2. Second order equations.

(a) Solve the initial value problem.

$$y'' + 3y' = 0, \qquad y(0) = -2, \qquad y'(0) = 3 \tag{6}$$

$r^2 + 3r = 0$, $r(r+3) = 0$

$r_1 = 0$, $r_2 = -3$

$y = c_1 + c_2 e^{-3t}$ $\quad y(0) = c_1 + c_2 = -2$

$y' = -3c_2 e^{-3t}$ $\quad y'(0) = -3c_2 = 3$

so $c_2 = -1$ and $c_1 = -1$

$y = -1 - e^{-3t}$

(b) Find the Wronskian of the functions

$$y_1(t) = e^{-2t}, \qquad y_2(t) = te^{-2t} \tag{7}$$

$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (e^{-2t} - 2te^{-2t}) \end{vmatrix}$

$= e^{-2t}(e^{-2t} - 2te^{-2t}) - te^{-2t}(-2e^{-2t})$

$= e^{-4t} - 2te^{-4t} + 2te^{-4t} = e^{-4t}$

(c) Use Euler's formula for complex exponentials to write e^{2-3i} in the form $a + bi$.

$$e^{2-3i} = e^2 e^{-3i} = e^2(\cos(-3) + i\sin(-3))$$
$$= e^2(\cos 3 - i\sin 3)$$
$$= (e^2\cos 3) + (-e^2\sin 3)\,i$$

(d) Find the general solution.

$$y'' - 2y' + 6y = 0 \tag{8}$$

$$r^2 - 2r + 6 = 0 \Rightarrow r = \frac{2 \pm \sqrt{4 - 4\cdot 6}}{2}$$

$$r = \frac{2 \pm \sqrt{-20}}{2} = \frac{2 \pm \sqrt{4}\sqrt{5}\,i}{2} = 1 \pm \sqrt{5}\,i$$

$$y = c_1 e^t \cos\sqrt{5}t + c_2 e^t \sin\sqrt{5}t$$

(e) Find the general solution.

$$4y'' - 4y' - 3y = 0 \tag{9}$$

$$4r^2 - 4r - 3 = 0 \Rightarrow r = \frac{4 \pm \sqrt{16 + 4\cdot 3\cdot 4}}{2\cdot 4}$$

$$r = \frac{4 \pm \sqrt{64}}{2\cdot 4} = \frac{4 \pm 8}{8} = \frac{1}{2} \pm 1 = \begin{cases} 3/2 \\ -1/2 \end{cases}$$

$$y = c_1 e^{(3/2)t} + c_2 e^{-(1/2)t}$$

(f) Find the general solution. Hint: undetermined coefficients.

$$y'' - 2y' - 3y = -3te^{-t} \tag{10}$$

$$r^2 - 2r - 3 = 0$$

$$(r+1)(r-3) = 0$$

since $r = -1$ is a root, we'll need extra powers of t

Try $y = (At^2 + Bt)e^{-t}$

$$y' = (2At + B)e^{-t} - (At^2 + Bt)e^{-t} = (-At^2 + (2A-B)t + B)e^{-t}$$

$$y'' = (-2At + 2A - B)e^{-t} - (-At^2 + (2A-B)t + B)e^{-t}$$

$$= (At^2 + (B-4A)t + 2A - 2B)e^{-t}$$

$$y'' - 2y' - 3y = (At^2 + (B-4A)t + 2A - 2B + 2At^2 - 2(2A-B)t - 2B$$
$$-3At^2 - 3Bt)e^{-t}$$

$$= (0t^2 + (B - 4A - 4A + 2B - 3B)t + (2A - 2B - 2B))e^{-t}$$

$$= (-8At + (2A - 4B))e^{-t}$$

$$\Rightarrow -8A = -3 \Rightarrow A = \frac{3}{8}$$

$$2A - 4B = 0 \Rightarrow 2B = A \Rightarrow B = \frac{3}{16}$$

Particular:

$$y = \left(\frac{3}{8}t^2 + \frac{3}{16}t\right)e^{-t}$$

general

$$\boxed{y = c_1 e^{-t} + c_2 e^{3t} + \left(\frac{3}{8}t^2 + \frac{3}{16}t\right)e^{-t}}$$

(g) Find a particular solution. Hint: variation of parameters.

$$y'' - 2y' + y = \frac{e^t}{1+t^2} \tag{11}$$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1$ repeated root

$y_1 = e^t \quad y_2 = te^t$

wronskian : $W(y_1, y_2) = \begin{vmatrix} e^t & te^t \\ e^t & (e^t + te^t) \end{vmatrix}$

$= e^t(e^t + te^t) - te^t e^t = e^{2t}$

$y_p = u_1 y_1 + u_2 y_2$

$u_1' = \frac{-y_2 g}{W} = \frac{-te^t e^t (1+t^2)^{-1}}{e^{2t}} = \frac{-t}{1+t^2}$

$u_1 = \int \frac{-t}{1+t^2} dt = -\frac{1}{2} \int \frac{2t\,dt}{1+t^2} = -\frac{1}{2} \ln(1+t^2)$

$u_2' = \frac{y_1 g}{W} = \frac{e^t e^t (1+t^2)^{-1}}{e^{2t}} = \frac{1}{1+t^2}$

$u_2 = \int \frac{1}{1+t^2} dt = \arctan t$

$y_p = -\frac{1}{2} \ln(1+t^2) e^t + (\arctan t) te^t$

3. Higher order equations.

(a) Determine whether these 4 functions are linearly independent:

$$f_1(t) = 2t - 3, \quad f_2(t) = t^3 + 1, \quad f_3(t) = 2t^2 - t, \quad f_4(t) = t^2 + t + 1 \tag{12}$$

$0 = k_1 f_1 + k_2 f_2 + k_3 f_3 + k_4 f_4$

$= 2k_1 t - 3k_1 + k_2 t^3 + k_2 + 2k_3 t^2 - k_3 t + k_4 t^2 + k_4 t + k_4$

$= (k_2) t^3 + (2k_3 + k_4) t^2 + (2k_1 - k_3 + k_4) t + (-3k_1 + k_2 + k_4)$

$\Rightarrow k_2 = 0$

$2k_3 + k_4 = 0 \Rightarrow k_4 = -2k_3$

$2k_1 - k_3 + k_4 = 0 \rightarrow 2k_1 - k_3 - 2k_3 = 0 \rightarrow 2k_1 - 3k_3 = 0$

$k_1 = \frac{3}{2} k_3$

$-3k_1 + k_2 + k_4 = 0$

$-3k_1 - 2k_3 = 0 \rightarrow -3k_1 = 2k_3$

$k_1 = -\frac{2}{3} k_3 \longrightarrow$ together imply

$k_3 = k_1 = 0$

so $k_4 = 0$ as well

so the only relation is when all $k_1 = k_2 = k_3 = k_4 = 0$

Therefore these functions are linearly independent.

(b) Find the general solution of the eigth-order equation

$$y^{(8)} + 8y^{(4)} + 16y = 0 \tag{13}$$

$$r^8 + 8r^4 + 16 = 0$$

$s = r^4$ $\quad s^2 + 8s + 16 = 0$

$$(s+4)^2 = 0$$

$$(r^4+4)^2 = 0$$

$$r^4 = -4$$

$$r^2 = \pm 2i$$

$r^2 = 2i$ use $\sqrt{i} = \pm e^{(\pi/4)i} = \pm\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)$

$$r = \pm(1+i)$$

$r^2 = -2i$ use $\sqrt{-i} = \pm\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right)$

$$r = \pm(1-i)$$

Thus there are 4 roots, each of which is repeated

$\left.\begin{array}{l} 1+i \Leftrightarrow 1-i \\ -1+i \Leftrightarrow -1-i \end{array}\right\}$ conjugate pairs

$$y = c_1 e^t \cos t + c_2 e^t \sin t + c_3 e^{-t} \cos t + c_4 e^{-t} \sin t$$
$$c_5 t e^t \cos t + c_6 t e^t \sin t + c_7 t e^{-t} \cos t + c_8 t e^{-t} \sin t$$

4. Power series.

(a) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \tag{14}$$

ratio test: $L = \lim_{n\to\infty} \frac{\left|\frac{x^{2(n+1)}}{(n+1)!}\right|}{\left|\frac{x^{2n}}{n!}\right|} = \lim_{n\to\infty} \frac{|x|^{2n+2}}{|x|^{2n}} \frac{n!}{(n+1)!}$

$= \lim_{n\to\infty} |x|^2 \frac{1}{n+1} = 0$

since $L = 0 < 1$, the series converges for any value of x: Radius of convergence $= \infty$.

(b) Re-index this series so that the general term involves x^n:

$$x \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n \tag{15}$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=1}^{\infty} (n+1)(n)\, a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=\boxed{0}}^{\infty} (n+1)(n)\, a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n$$

since the n=0 term is 0.

$$= \sum_{n=0}^{\infty} \left[(n+1)n\, a_{n+1} + a_n\right] x^n$$

(c) Seek a power series solution of the following equation at the point $x_0 = 0$:

$$(1+x^2)y'' - 4xy' + 6y = 0 \tag{16}$$

Find the recurrence relation, and determine the first four terms of the solution that begins with $a_0 = 1$, $a_1 = 1$.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$xy' = x\sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^n$$

$$(1+x^2)y'' = (1+x^2)\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}$$

$$= \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n x^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n + \sum_{n=0}^{\infty} n(n-1)a_n x^n$$

$$0 = \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + n(n-1)a_n - 4na_n + 6a_n\right] x^n$$

Recurrence Relation: $(n+2)(n+1)a_{n+2} + [n(n-1) - 4n + 6]\, a_n = 0$

$$a_{n+2} = \frac{-(n^2 - 5n + 6)\, a_n}{(n+2)(n+1)}$$

$a_0 = 1$

$a_1 = 1$

$$a_2 = \frac{-6}{2\cdot 1} a_0 = -3a_n = -3$$

$$a_3 = \frac{-(1-5+6)}{3\cdot 2} a_1 = \frac{-2}{6} = -\frac{1}{3}$$

$$y = 1 + x - 3x^2 - \frac{1}{3}x^3 + \cdots$$

(d) Find the general solution for $x > 0$ of the differential equation:

$$2x^2y'' - 4xy' + 6y = 0 \tag{17}$$

Note: this equation is an Euler equation with a singular point at $x_0 = 0$.

$y = x^r \quad xy' = rx^r \quad x^2y'' = r(r-1)x^r$

$2r(r-1)x^r - 4rx^r + 6x^r = 0$

$2r(r-1) - 4r + 6 = 0$

$2r^2 - 2r - 4r + 6 = 0$

$2r^2 - 6r + 6 = 0$

$r^2 - 3r + 3 = 0$

$\Rightarrow r = \dfrac{3 \pm \sqrt{9-12}}{2} = \dfrac{3 \pm \sqrt{3}\, i}{2}$

$x^{\frac{3 \pm \sqrt{3} i}{2}} = x^{3/2}\left(\cos\left(\frac{\sqrt{3}}{2}\ln x\right) \pm i\sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right)$

General solution:

$y = c_1 x^{3/2}\cos\left(\frac{\sqrt{3}}{2}\ln x\right) + c_2 x^{3/2}\sin\left(\frac{\sqrt{3}}{2}\ln x\right)$

5. Laplace transform. (A copy of the table on p. 317 of the textbook will be provided during the exam.)

(a) Here are some functions of s. Find their inverse Laplace transforms. The answer may have discontinuities or delta functions in it.

$$F(s) = \frac{2s-3}{s^2+2s+10} \tag{18}$$

$$s^2+2s+10 = (s+1)^2 + 9 = (s+1)^2 + 3^2$$

$$\mathcal{L}\{e^{-t}\sin 3t\} = \frac{3}{(s+1)^2+9} \qquad \mathcal{L}\{e^{-t}\cos 3t\} = \frac{s+1}{(s+1)^2+9}$$

$$2s-3 = 2(s+1) - 5 = 2(s+1) - \frac{5}{3}\cdot 3$$

$$f(t) = 2e^{-t}\cos 3t - \frac{5}{3}e^{-t}\sin 3t$$

$$F(s) = \frac{(s-2)e^{-s}}{s^2-4s+3} \tag{19}$$

$$s^2-4s+3 = (s-1)(s-3)$$

$$\frac{s-2}{(s-1)(s-3)} = \frac{A}{s-1} + \frac{B}{s-3} = \frac{1/2}{s-1} + \frac{1/2}{s-3} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{2}e^{t} + \frac{1}{2}e^{3t}$$

$$(s-2) = A(s-3) + B(s-1)$$

$s=3$: $1 = 2B$ $\quad B = \frac{1}{2}$

$s=1$: $-1 = A(-2)$ $\quad A = \frac{1}{2}$

$$\text{so } f(t) = u_1(t)\left[\frac{1}{2}e^{(t-1)} + \frac{1}{2}e^{3(t-1)}\right]$$

$$F(s) = \frac{1}{(s+1)^2(s^2+4)} \tag{20}$$

$$F(s) = G(s)H(s) \qquad G(s) = \frac{1}{(s+1)^2} \Leftrightarrow g(t) = te^{-t}$$

$$H(s) = \frac{1}{s^2+4} = \frac{1}{2}\,\frac{2}{s^2+2^2} \Leftrightarrow h(t) = \frac{1}{2}\sin(2t)$$

$$f(t) = \int_0^t g(\tau)\,h(t-\tau)\,d\tau = \int_0^t \tau e^{-\tau}\cdot\frac{1}{2}\cdot\sin[2(t-\tau)]\,d\tau$$

(b) For each of following equations, solve for $Y(s) = \mathcal{L}\{y(t)\}$.

$$y'' - 4y' + 4y = 0, \quad y(0) = 10, \quad y'(0) = 5 \tag{21}$$

$$\left[s^2Y(s) - sy(0) - y'(0)\right] - 4\left[sY(s) - y(0)\right] + 4Y(s) = 0$$

$$(s^2 - 4s + 4)Y(s) - 10s - 5 - 4(-10) = 0$$

$$(s^2 - 4s + 4)Y(s) - 10s + 35 = 0 \Rightarrow Y(s) = \frac{10s - 35}{s^2 - 4s + 4}$$

$$y'' - 2y' + 2y = \sin(100t), \quad y(0) = -1, \quad y'(0) = 1 \tag{22}$$

$$\left[s^2Y(s) - s(-1) - (1)\right] - 2\left[sY(s) - (-1)\right] + 2Y(s) = \frac{100}{s^2 + 100^2}$$

$$(s^2 - 2s + 2)Y(s) + s - 1 - 2 = \frac{100}{s^2 + 100^2}$$

$$Y(s) = \frac{1}{s^2 - 2s + 2}\left[-s + 3 + \frac{100}{s^2 + 100^2}\right]$$

$$y'' + 4y = u_3(t) - u_4(t), \quad y(0) = 0, \quad y'(0) = 0 \tag{23}$$

$$(s^2 + 4)Y(s) = \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s}$$

$$Y(s) = \frac{e^{-3s} - e^{-4s}}{s(s^2 + 4)}$$

$$y'' + 2y' + 2y = \delta(t - \pi) + \delta(t - 2\pi) + \delta(t - 3\pi), \quad y(0) = 0, \quad y'(0) = 0 \tag{24}$$

$$(s^2 + 2s + 2)Y(s) = e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s}$$

$$Y(s) = \frac{e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s}}{s^2 + 2s + 2}$$

6. First order systems.

(a) In each of these two cases, find the eigenvalues and eigenvectors of the matrix $\mathbf{A}$, and find the general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \tag{25}$$

$$\begin{vmatrix} 1-r & 1 \\ 4 & -2-r \end{vmatrix} = (1-r)(-2-r)-4 = -2-r+2r+r^2-4$$

$$= r^2+r-6$$

$$r = \frac{-1 \pm \sqrt{1+4\cdot 6}}{2} = \frac{-1\pm 5}{2} = 2 \text{ or } -3$$

$$r_1 = 2: \begin{pmatrix} 1-2 & 1 \\ 4 & -2-2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}$$

$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector

$$r_2 = -3 \quad \begin{pmatrix} 1+3 & 1 \\ 4 & -2+3 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}$$

$v_2 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ is one eigenvector

$$y = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 4 \end{pmatrix} e^{-3t}$$

$$\mathbf{A} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \tag{26}$$

$$\begin{vmatrix} 2-r & -5 \\ 1 & -2-r \end{vmatrix} = (2-r)(-2-r)+5 = -4+r^2+5$$

$$= r^2+1 \quad \Rightarrow \quad r = \pm i$$

$r_1 = i$: $\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}$ $\quad V_1 = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$ is an eigenvector

$r_2 = -i$ $\begin{pmatrix} 2+i & -5 \\ 1 & -2+i \end{pmatrix}$ $\quad V_2 = \begin{pmatrix} 5 \\ 2+i \end{pmatrix}$ is an eigenvector

complex solution $e^{it}\begin{pmatrix} 5 \\ 2-i \end{pmatrix} = \begin{pmatrix} 5\cos t + i\,5\sin t \\ 2\cos t + 2i\sin t - i\cos t + \sin t \end{pmatrix}$

real part: $\begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix}$ $\quad$ Imaginary part: $\begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$

general real solution : $y = c_1\begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2\begin{pmatrix} 5\sin t \\ 2\sin t - \cos t \end{pmatrix}$

7. Fourier series and partial differential equations

(a) Either solve the boundary value problem or show that it has no solution.

$$y'' + 2y = 0, \qquad y'(0) = 1, \qquad y'(\pi) = 0 \tag{27}$$

$$y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x$$

$$y' = -\sqrt{2}c_1 \sin\sqrt{2}x + \sqrt{2}c_2 \cos\sqrt{2}x$$

$$1 = y'(0) = -\sqrt{2}c_1 \sin 0 + \sqrt{2}c_2 \cos 0 = \sqrt{2}c_2 \Rightarrow c_2 = \frac{1}{\sqrt{2}}$$

$$0 = y'(\pi) = -\sqrt{2}c_1 \sin\sqrt{2}\pi + \cos\sqrt{2}\pi$$

$$c_1 = \frac{1}{\sqrt{2}} \frac{\cos\sqrt{2}\pi}{\sin\sqrt{2}\pi} = \frac{1}{\sqrt{2}} \cot\sqrt{2}\pi$$

$$y = \frac{1}{\sqrt{2}}\left(\cot\sqrt{2}\pi\right)\cos\sqrt{2}x + \frac{1}{\sqrt{2}}\sin\sqrt{2}x$$

(b) Find the eigenvalues and eigenfunctions of the boundary value problem. Assume all the eigenvalues are real.

$$y'' + \lambda y = 0, \qquad y'(0) = 0, \qquad y(\pi) = 0 \tag{28}$$

$\lambda > 0$: let $\lambda = \mu^2$ $\quad y = c_1 \cos\mu x + c_2 \sin\mu x$

$y' = -\mu c_1 \sin\mu x + \mu c_2 \cos\mu x$

$0 = y'(0) = \mu c_2 \Rightarrow c_2 = 0 \qquad y = c_1 \cos\mu x$

$0 = y(\pi) = c_1 \cos\mu\pi \Rightarrow$ need $\cos\mu\pi = 0 \Rightarrow \mu = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$

so $\lambda = \mu^2$ is $\left(\frac{1}{2}\right)^2, \left(\frac{3}{2}\right)^2, \left(\frac{5}{2}\right)^2, \ldots$

y is $\cos\frac{1}{2}x, \cos\frac{3}{2}x, \cos\frac{5}{2}x, \ldots$

} these are the eigenvalues and eigenfunctions with $\lambda > 0$

$\lambda = 0$: $y = c_1 + c_2 x \qquad y' = c_2 \qquad y'(0) = 0 \Rightarrow c_2 = 0$

$y(\pi) = 0 \Rightarrow c_1 = 0$ $\quad$ so $y = 0$. $\lambda = 0$ is not an eigenvalue

$\lambda < 0$ $\quad \lambda = -\mu^2 \qquad y = c_1 e^{\mu x} + c_2 e^{-\mu x}$

$y' = \mu c_1 e^{\mu x} - \mu c_2 e^{-\mu x}$

$0 = y'(0) = \mu c_1 - \mu c_2 \Rightarrow c_1 = c_2 \Rightarrow y = c\left(e^{\mu x} + e^{-\mu x}\right)$

$0 = y(\pi) = c\left(e^{\mu\pi} + e^{-\mu\pi}\right)$

But $e^{\mu\pi} + e^{-\mu\pi} \neq 0$ since both terms are positive $\Rightarrow c = 0$

So $\lambda < 0$ is not an eigenvalue

(c) Find the fundamental period of the function defined by

$$f(x) = \begin{cases} (-1)^n, & 2n-1 \le x < 2n \\ 1, & 2n \le x < 2n+1 \end{cases} \tag{29}$$

In this definition, n ranges over all integers.

$f(x) = 1 \quad 0 < x < 1$

$-1 = (-1)^1 \quad 1 < x < 2$

$1 \quad 2 < x < 3$

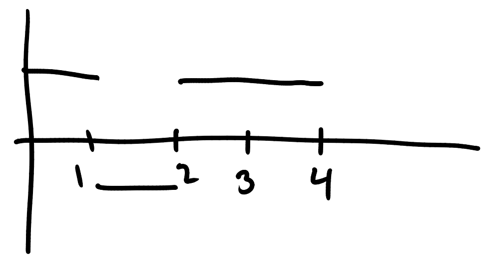
$1 = (-1)^2 \quad 3 < x < 4$

$1 \quad 4 < x < 5$

$-1 = (-1)^3 \quad 5 < x < 6$

$1 \quad 6 < x < 7$

$1 = (-1)^4 \quad 7 < x < 8$



$\boxed{\text{Fundamental period} = 4}$

(d) Find the Fourier series of the periodic function with period $2L$ defined on the interval $-L \le x < L$ by

$$f(x) = \begin{cases} 1, & -L \le x < 0 \\ 0, & 0 \le x < L \end{cases} \tag{30}$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{-L}^{0} 1\,dx = \frac{1}{L} L = 1$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} \cos\frac{m\pi x}{L}\,dx$$

$$= \frac{1}{L}\left[\frac{L}{m\pi}\sin\frac{m\pi x}{L}\right]_{-L}^{0} = \frac{1}{L}\frac{L}{m\pi}\left[0 - \sin(-m\pi)\right] = 0$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} \sin\frac{m\pi x}{L}\,dx$$

$$= \frac{1}{L}\left[\frac{L}{m\pi}(-1)\cos\frac{m\pi x}{L}\right]_{-L}^{0} = \frac{1}{L}\frac{L}{m\pi}\left[-1 + \cos -m\pi\right]$$

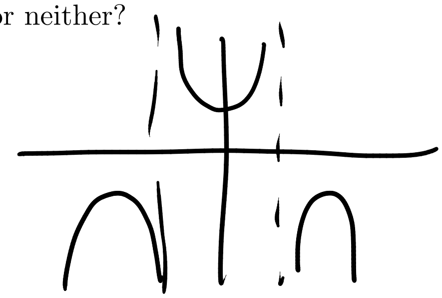
$$= \frac{1}{m\pi}\left[-1 + \cos m\pi\right] = \frac{1}{m\pi}\left[-1 + (-1)^m\right] = \begin{cases} 0 & \text{if } m \text{ even} \\ \frac{-2}{m\pi} & \text{if } m \text{ odd} \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}$$

$$= \frac{1}{2} + \sum_{m \text{ odd}} \frac{-2}{m\pi}\sin\frac{m\pi x}{L} = \frac{1}{2} - \frac{2}{\pi}\sum_{m \text{ odd}} \frac{1}{m}\sin\frac{m\pi x}{L}$$

(e) Is $f(x) = \sec x$ even, odd, or neither?

$$\sec x = \frac{1}{\cos x}$$



is <u>even</u>.

(f) Find the sine series (with period 6π) of the function

$$L = 3\pi \qquad f(x) = \begin{cases} 0, & 0 < x < \pi \\ 1, & \pi < x < 2\pi \\ 2, & 2\pi < x < 3\pi \end{cases} \tag{31}$$

$$f(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{3\pi} = \sum_{n=1}^{\infty} c_n \sin \frac{nx}{3}$$

$$C_n = \frac{2}{3\pi} \int_0^{3\pi} f(x) \sin \frac{nx}{3}\, dx = \frac{2}{3\pi} \left[\int_{\pi}^{2\pi} \sin \frac{nx}{3}\, dx + \int_{2\pi}^{3\pi} 2 \sin \frac{nx}{3}\, dx \right]$$

$$= \frac{2}{3\pi} \left\{ \left[-\frac{3}{n} \cos \frac{nx}{3} \right]_{\pi}^{2\pi} + 2 \left[-\frac{3}{n} \cos \frac{nx}{3} \right]_{2\pi}^{3\pi} \right\}$$

$$= \frac{2}{3\pi} \frac{3}{n} \left\{ -\cos \frac{2\pi n}{3} + \cos \frac{n\pi}{3} - 2 \cos \frac{3\pi n}{3} + 2 \cos \frac{2\pi n}{3} \right\}$$

$$= \frac{2}{n\pi} \left\{ \cos \frac{\pi n}{3} + \cos \frac{2\pi n}{3} - 2 \cos \pi n \right\}$$

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \cos \frac{\pi n}{3} + \cos \frac{2\pi n}{3} - 2 \cos \pi n \right\} \sin \frac{nx}{3}$$

(g) Find the solution of the heat equation

$$\frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2} \tag{32}$$

on the interval $0 < x < \pi$, with boundary conditions $u(0,t) = 0$, $u(\pi,t) = 0$, and initial temperature distribution

$$u(x,0) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin nx \tag{33}$$

General solution $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin \frac{n\pi x}{L}$

Here $\alpha^2 = 9$ and $L = \pi$

$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-9n^2 t} \sin nx$

$u(x,0) = \sum_{n=1}^{\infty} c_n \sin nx \overset{?}{=} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin nx$ Take $c_n = \frac{(-1)^n}{n^2}$

So solution is $u(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} e^{-9n^2 t} \sin nx$

(h) Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 100 \frac{\partial^2 u}{\partial x^2} \tag{34}$$

on the interval $0 < x < L$, with the boundary conditions $u(0, t) = 0$, $u(L, t) = 0$, and the initial conditions

$$u(x, 0) = \sin \frac{5\pi x}{L}, \qquad \frac{\partial u}{\partial t}(x, 0) = 0 \tag{35}$$

General solution $u(x,t) = \sum_{n=1}^{\infty} c_n \cos \frac{n\pi a t}{L} \sin \frac{n\pi x}{L}$

Here $a^2 = 100$ so $a = 10$

$u(x,0) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L} \stackrel{?}{=} \sin \frac{5\pi x}{L}$

Take $c_5 = 1$ all other $c_n = 0$

$u(x,t) = \cos \frac{50\pi t}{L} \sin \frac{5\pi x}{L}$