NAME: SOLUTIONS

EID:

M 427K Final Exam Version B December 14, 2012 Instructor: James Pascaleff

Problem	Possible	Actual
1	20	
2	20	
3	10	
4	15	
5	10	
6	15	
7	25	
8	20	
9	20	
10	20	
11	25	
Total	200	

INSTRUCTIONS:

- Do all work on these sheets.
- Show all work.
- A table of Laplace transforms is provided.
- In addition, you are permitted one two-sided sheet of notes (8.5" by 11"). No other notes, books, calculators, or other electronic devices are permitted.

First Order Equations

1. (20 points) Consider the autonomous differential equation

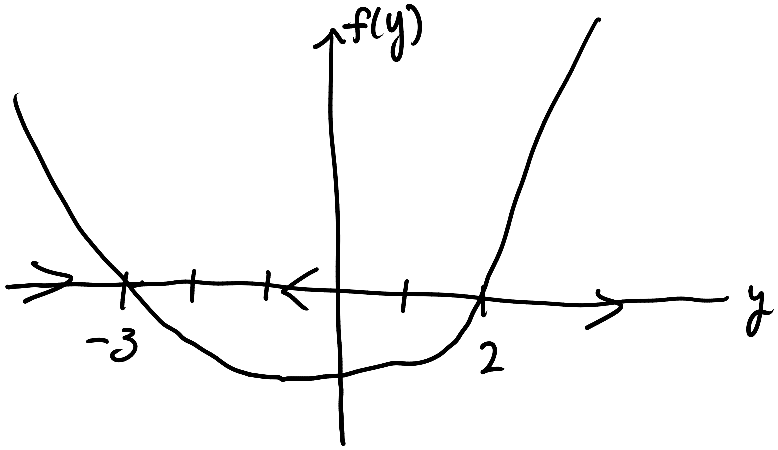
$$\frac{dy}{dt} = (y-2)(y+3)$$

(a) Find the critical (equilibrium) points, and classify each as stable, unstable, or semistable.

$f(y) = (y-2)(y+3)$

critical: $f(y) = 0$

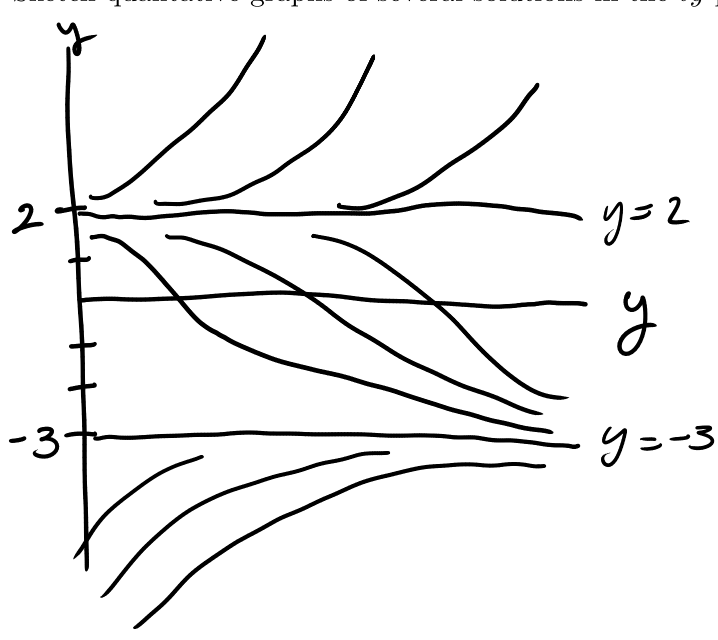
$y = 2$ or $y = -3$



$y = 2$ is unstable

$y = -3$ is stable

(b) Sketch qualitative graphs of several solutions in the ty-plane.



2. (20 points) Find the general solution.

$$y' - 4y = t^2 e^{4t}$$

integrating factor $\mu = e^{\int -4dt} = e^{-4t}$

multiply by μ:

$$e^{-4t}y' - 4e^{-4t}y = t^2$$

$$(e^{-4t}y)' = t^2$$

$$e^{-4t}y = \int t^2 dt = \frac{1}{3}t^3 + C$$

$$y = \frac{1}{3}t^3 e^{4t} + Ce^{4t}$$

3. (10 points) Use one step of Euler's method to approximate $y(0.1)$, where $y(0) = 1$, and y satisfies

$$y' = y - 2t$$

One step from $t=0$ to $t=0.1$, use $h=0.1$

$y'(0) = y(0) - 2(0) = 1$

t	y	y'	h	$y' \cdot h$
0	1	1	0.1	0.1
0.1	1.1			

$\boxed{y(0.1) \approx 1.1}$

Second Order Linear Equations

4. (15 points) Assuming s and t are real numbers, write $e^{s+it} - e^{s-it}$ in the form $a + ib$ where a and b are real numbers.

$e^{s+it} = e^s e^{it} = e^s \cos t + i e^s \sin t$

$e^{s-it} = e^s e^{-it} = e^s \cos(-t) + i e^s \sin(-t)$

$= e^s \cos t - i e^s \sin t$

$e^{s+it} - e^{s-it} = (e^s \cos t + i e^s \sin t) - (e^s \cos t - i e^s \sin t)$

$= 2i e^s \sin t$

5. (10 points) Write and solve the characteristic equation, and find the general solution of the differential equation.

$$y'' - 9y' = 0$$

$r^2 - 9r = 0$

$r(r-9) = 0$

$r_1 = 0 \quad r_2 = 9$

$$y = c_1 e^{0t} + c_2 e^{9t}$$
$$= c_1 + c_2 e^{9t}$$

6. (15 points) Use the method of variation of parameters to find a particular solution to this nonhomogeneous equation. It is acceptable to leave your answer in terms of one or more indefinite integrals, as long as the integrand (the function you are integrating) is written out explicitly as a function of t. You do not need to evaluate the integral. For example: $\cos(t) \int (\ln(t))^2 \, dt$ would be an acceptable form.

$$y'' - 9y' = \frac{\cos(t)}{1 + e^t}$$

solutions of homogeneous equation from #5 $\begin{cases} y_1 = e^{0t} = 1 \\ y_2 = e^{9t} \end{cases}$ $\quad y_1' = 0$, $\quad y_2' = 9e^{9t}$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^{9t} \\ 0 & 9e^{9t} \end{vmatrix} = 9e^{9t}$$

Solution is $y = u_1 y_1 + u_2 y_2$ when

$$u_1' = \frac{-y_2 g}{W} = \frac{-e^{9t}\left(\frac{\cos t}{1+e^t}\right)}{9e^{9t}} = -\frac{1}{9}\left(\frac{\cos t}{1+e^t}\right)$$

$$u_2' = \frac{y_1 g}{W} = \frac{1\left(\frac{\cos t}{1+e^t}\right)}{9e^{9t}} = \frac{1}{9} e^{-9t}\left(\frac{\cos t}{1+e^t}\right)$$

$$y = 1 \int -\frac{1}{9}\left(\frac{\cos t}{1+e^t}\right) dt + e^{9t} \int \frac{1}{9} e^{-9t}\left(\frac{\cos t}{1+e^t}\right) dt$$

7. (25 points) Using Laplace transform methods, solve the initial value problem.

$$y'' + 16y = \delta(t - 3\pi), \qquad y(0) = 2, \qquad y'(0) = 1$$

Let $Y(s) = \mathcal{L}\{y(t)\}$

$\mathcal{L}\{y''\} = s^2 Y(s) - sy(0) - y'(0) = s^2 Y(s) - 2s - 1$

$\mathcal{L}\{\delta(t - 3\pi)\} = e^{-3\pi s}$

$(s^2+16)Y(s) - 2s - 1 = e^{-3\pi s}$

$$Y(s) = \frac{2s}{s^2+16} + \frac{1}{s^2+16} + \frac{e^{-3\pi s}}{s^2+16}$$

$$= 2\left(\frac{s}{s^2+16}\right) + \frac{1}{4}\left(\frac{4}{s^2+16}\right) + \frac{1}{4}\left(\frac{4}{s^2+16}\right)e^{-3\pi s}$$

Take $\mathcal{L}^{-1}$:

$$y(t) = 2\cos 4t + \frac{1}{4}\sin 4t + \frac{1}{4} u_{3\pi}(t)\sin[4(t-3\pi)]$$

8. (20 points) Find the eigenvalues and eigenvectors of the matrix A, and find the general solution to the equation $\mathbf{x}' = A\mathbf{x}$.

$$A = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$$

$$A - rI = \begin{pmatrix} 2-r & 2 \\ -1 & -1-r \end{pmatrix}$$

$$\det(A - rI) = (2-r)(-1-r) + 2 = -2 + r - 2r + r^2 + 2$$
$$= r^2 - r = r(r-1)$$

Eigenvalues: $r_1 = 0 \quad r_2 = 1$

$\boxed{r_1 = 0}$ $\begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector

(any constant multiple is also an eigenvector)

$\boxed{r_2 = 1}$ $\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $v_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is an eigenvector

General solution: $\vec{x}(t) = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^t \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

9. (20 points) Consider the eigenvalue boundary value problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, \qquad y'(5) = 0$$

Is $\lambda = 16\pi^2$ an eigenvalue of this problem? If it is, find a corresponding eigenfunction. If not, show why no such eigenfunction exists.

$y'' + 16\pi^2 y = 0 \Rightarrow y = c_1 \cos 4\pi x + c_2 \sin 4\pi x$

$y' = -4\pi c_1 \sin 4\pi x + 4\pi c_2 \cos 4\pi x$

$y(0) = 0 \Rightarrow c_1 = 0$

$y'(5) = 0 \Rightarrow 4\pi c_2 \cos 4\pi \cdot 5 = 0 \Rightarrow 4\pi c_2 \cos 20\pi = 0$

since $\cos 20\pi = 1$ $\quad 4\pi c_2 = 0 \Rightarrow c_2 = 0$

only solution is $y = 0$: $\lambda = 16\pi^2$ is not an eigenvalue

Answer the same question for $\lambda = \pi^2/4$.

$y'' + \frac{\pi^2}{4} y = 0 \Rightarrow y = c_1 \cos \frac{\pi}{2} x + c_2 \sin \frac{\pi}{2} x$

$y' = -\frac{\pi}{2} c_1 \sin \frac{\pi}{2} x + \frac{\pi}{2} c_2 \cos \frac{\pi}{2} x$

$y(0) = 0 \Rightarrow c_1 = 0$

$y'(5) = 0 \Rightarrow \frac{\pi}{2} c_2 \cos \frac{\pi}{2} 5 = 0$

sin $\cos \frac{5\pi}{2} = 0$, there is no condition on c_2:

$y = c_2 \sin \frac{\pi}{2} x$ is an eigenfunction, and $\lambda = \pi^2/4$ is an eigenvalue.

10. (20 points) This question is about separation of variables in a partial differential equation. *The problem here is slightly different from any of the problems we considered in lecture, so read carefully.* Consider the heat conduction problem on the domain $0 < x < \pi$ and $t > 0$

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$$

with boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(\pi,t) = 0$$

and initial temperature distribution

$$u(x,0) = \cos 3x$$

It turns out this problem has a separable solution of the form $u(x,t) = X(x)T(t)$. Find it by following these steps.

(a) Plug $t = 0$ into the separable solution, and assume $T(0) = 1$ in order to determine $X(x)$.

see version A solutions

(b) Find the ordinary differential equation that $T(t)$ satisfies.

(c) Solve that equation to determine $T(t)$, and put everything together to get $u(x,t)$.

11. (25 points) Consider the function that is periodic with period 6 and defined on the interval $-3 \le x < 3$ by

$$f(x) = \begin{cases} 2, & 0 \le x < 3 \\ -2, & -3 \le x < 0 \end{cases}$$

Find the Fourier series for this function. Hint: There is an easy way to see that a lot of the coefficients of the Fourier series are zero. (Though you still need to compute all the ones that aren't zero!)

Since $f(x)$ is an odd function, the constant term and the cosine coefficients are zero. Also, the sine coefficients are

$$b_n = \frac{1}{3}\int_{-3}^{3} f(x)\sin\frac{n\pi x}{3}\,dx = \frac{2}{3}\int_0^3 f(x)\sin\frac{n\pi x}{3}\,dx$$

$$= \frac{2}{3}\int_0^3 2\sin\frac{n\pi x}{3}\,dx = \frac{4}{3}\left[-\frac{3}{n\pi}\cos\frac{n\pi x}{3}\right]_0^3$$

$$= \frac{-4}{n\pi}\left[\cos n\pi - 1\right] = \frac{-4}{n\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{8}{n\pi} & n \text{ odd} \end{cases}$$

$$f(x) = \frac{8}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin\frac{n\pi x}{3}$$