NAME: Solutions

EID:

M 427K Exam 2 Version B   November 13, 2012   Instructor: James Pascaleff

Problem	Possible	Actual
1	10	
2	7	
3	20	
4	10	
5	5	
6	15	
7	18	
8	15	
Total	100	

INSTRUCTIONS:

- Do all work on these sheets.
- Show all work.
- No books, notes, calculators, or other electronic devices.

Higher order equations

1. (10 points) Find the general solution of the fourth order equation

$$y^{(4)} - 5y'' + 6y = 0$$

$$0 = r^4 - 5r^2 + 6 = (r^2-3)(r^2-2) = (r-\sqrt{3})(r+\sqrt{3})(r-\sqrt{2})(r+\sqrt{2})$$

roots $r_1 = \sqrt{3}$, $r_2 = -\sqrt{3}$, $r_3 = \sqrt{2}$, $r_4 = -\sqrt{2}$

$$y = c_1 e^{\sqrt{3}t} + c_2 e^{-\sqrt{3}t} + c_3 e^{\sqrt{2}t} + c_4 e^{-\sqrt{2}t}$$

Power series

2. (7 points) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$$

$$L = \lim_{n\to\infty} \frac{\left|\frac{2^{n+1}x^{n+1}}{(n+1)!}\right|}{\left|\frac{2^n x^n}{n!}\right|} = \lim_{n\to\infty} \frac{n!}{(n+1)!} \frac{2^{n+1}}{2^n} \frac{|x|^{n+1}}{|x|^n}$$

$$= \lim_{n\to\infty} \frac{1}{n+1} 2|x| = 0 < 1 \quad \text{for any } x.$$

So the radius of convergence is ∞; the series converges for all x.

3. (20 points) Seek a power series solution of the following equation at the point $x_0 = 0$:

$$y'' + 2xy' + y = 0$$

Find the recurrence relation, and determine the terms, up to the x^3 term, of the solution that begins with $a_0 = 0$, $a_1 = 1$.

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

$$xy' = x \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} n a_n x^n$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

$$0 = y'' + 2xy' + y = \sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + 2n a_n + a_n \right] x^n$$

$$\Rightarrow 0 = (n+2)(n+1) a_{n+2} + 2n a_n + a_n = (n+2)(n+1) a_{n+2} + (2n+1) a_n$$

$$\Rightarrow \quad a_{n+2} = \frac{-(2n+1) a_n}{(n+2)(n+1)}$$ is the recurrence relation

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = \frac{-1}{2 \cdot 1} a_0 = 0, \quad a_3 = \frac{-3}{3 \cdot 2} a_1 = \frac{-1}{2}$$

Solution is $y = 0 + 1 \cdot x + 0 \cdot x^2 - \frac{1}{2} x^3 + \cdots$

$$= x - \frac{1}{2} x^3 + \cdots$$

4. (10 points) The following two equations have a singular point at $x_0 = 0$. One of them has a regular singular point, while the other has an irregular singular point.

$$\text{Equation (A)} \quad x^2y'' + 3xy' + 5y = 0$$

$$\text{Equation (B)} \quad x^2y'' + 2y' + y = 0$$

(a) (4 points) Which equation has a regular singularity, and which has an irregular singularity? Circle your answer below.

(A) is regular, and (B) is irregular. OR (B) is regular, and (A) is irregular.

(b) (6 points) Justify your answer to the previous part using limits.

(A) is $y'' + \underbrace{\frac{3}{x}}_{p(x)} y' + \underbrace{\frac{5}{x^2}}_{q(x)} y = 0$

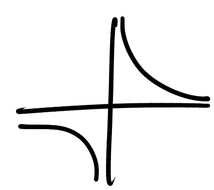
$\lim\limits_{x \to 0} x p(x) = \lim\limits_{x \to 0} 3 = 3$ exists and is finite

$\lim\limits_{x \to 0} x^2 q(x) = \lim\limits_{x \to 0} 5 = 5$ exists and is finite

So (A) is regular

(B) is $y'' + \underbrace{\frac{2}{x^2}}_{p(x)} y' + \underbrace{\frac{1}{x^2}}_{q(x)} y = 0$

$\lim\limits_{x \to 0} x p(x) = \lim\limits_{x \to 0} \frac{2}{x}$ does not exist



so (B) is irregular

Laplace transform

5. (5 points) Write the definition of the Laplace transform of a function $f(t)$.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

6. (15 points, 5 points per part) Here are some functions of s. Find their inverse Laplace transforms. The answer may have discontinuities or delta functions in it.

(a) $F(s) = \dfrac{2s-3}{s^2+2s+10}$

$$s^2+2s+10 = (s+1)^2+9 = (s+1)^2+3^2$$

$$F(s) = \frac{2s-3}{(s+1)^2+3^2} = \frac{2(s+1)-5}{(s+1)^2+3^2} = 2\frac{(s+1)}{(s+1)^2+3^2} - \frac{5}{3}\frac{3}{(s+1)^2+3^2}$$

$$f(t) = 2e^{-t}\cos 3t - \frac{5}{3}e^{-t}\sin 3t$$

(b) $F(s) = \dfrac{s(e^{-s}-e^{-2s})}{s^2+9}$ $= (e^{-s}-e^{-2s})\,G(s)$

where $G(s) = \dfrac{s}{s^2+9} = \mathcal{L}\{\cos 3t\}$ so $g(t) = \cos 3t$

$$f(t) = u_1(t)g(t-1) - u_2(t)g(t-2)$$
$$= u_1(t)\cos 3(t-1) - u_2(t)\cos 3(t-2)$$

(c) $F(s) = \dfrac{6G(s)}{s^4}$

For this part, $G(s) = \mathcal{L}\{g(t)\}$, and your answer will be in terms of $g(t)$.

$F(s) = G(s)H(s)$ where $H(s) = \dfrac{6}{s^4}$, so $h(t) = t^3$

$$f(t) = \int_0^t h(t-\tau)g(\tau)\,d\tau = \int_0^t (t-\tau)^3 g(\tau)\,d\tau$$

7. (18 points, 6 points per part) A mass on a spring with mass $m = 4$, damping $\gamma = 2$, spring constant $k = 2$, and subject to an external force $g(t)$ obeys the differential equation

$$4y'' + 2y' + 2y = g(t)$$

We impose the initial conditions $y(0) = 2$, $y'(0) = -1$. In each of the scenarios below, solve for $Y(s)$, the Laplace transform of $y(t)$. Do not take the inverse transform of your answer.

(a) First scenario: No forcing, $g(t) = 0$.

$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 2$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s + 1$$

So $4(s^2Y(s) - 2s + 1) + 2(sY(s) - 2) + 2Y(s) = 0$

$$(4s^2 + 2s + 2)Y(s) - 8s + 4 - 4 = 0$$

$$Y(s) = \frac{8s}{4s^2 + 2s + 2}$$

(b) Second scenario: The forcing function $g(t)$ consists of three delta–function impulses at times $t = \pi$, $t = 2\pi$, and $t = 3\pi$:

$$g(t) = \delta(t - \pi) + \delta(t - 2\pi) + \delta(t - 3\pi)$$

$$\mathcal{L}\{g(t)\} = e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s}$$

$$(4s^2 + 2s + 2)Y(s) - 8s = e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s}$$

$$Y(s) = \frac{8s}{4s^2 + 2s + 2} + \frac{(e^{-\pi s} + e^{-2\pi s} + e^{-3\pi s})}{4s^2 + 2s + 2}$$

(c) Third scenario: The forcing function $g(t)$ is zero up until time $t = \pi$, at which time it switches on to $\sin(t - \pi)$, or in symbols:

$$g(t) = \begin{cases} 0 & \text{if } t < \pi \\ \sin(t - \pi) & \text{if } t \geq \pi \end{cases}$$

$$g(t) = u_\pi(t) \sin(t - \pi)$$

$$\mathcal{L}\{g(t)\} = e^{-\pi s} \mathcal{L}\{\sin t\} = e^{-\pi s} \frac{1}{s^2+1}$$

$$(4s^2 + 2s + 2) Y(s) - 8s = e^{-\pi s} \frac{1}{s^2+1}$$

$$Y(s) = \frac{8s}{4s^2+2s+2} + \frac{e^{-\pi s}}{(4s^2+2s+2)(s^2+1)}$$

8. (15 points) Find the eigenvalues and eigenvectors of the matrix $\mathbf{A}$, and find the general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$
$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$$

$$\det(A - rI) = \det\begin{pmatrix} 1-r & 1 \\ 0 & -2-r \end{pmatrix} = (1-r)(-2-r) - 0\cdot 1$$
$$= (1-r)(-2-r)$$

roots $r_1 = 1$ $r_2 = -2$ are the eigenvalues.

Eigenvector for $r_1 = 1$: $\begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} ? \\ ? \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

can take $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, all eigenvectors are $c_1\vec{v}_1$ for const. c_1

Eigenvector for $r_2 = -2$: $\begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} ? \\ ? \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Can take $\vec{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, all eigenvectors are $c_2\vec{v}_2$ for const c_2

General solution of $\vec{x}' = A\vec{x}$:

$$\vec{x}(t) = c_1 e^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$