M 427K PRACTICE FOR SECOND MIDTERM EXAM

These problems are representative of the problems that will be on the second midterm exam. This list of problems is *longer* than the actual exam will be. All of these problems are similar to problems that were assigned for homework. A copy of the table of Laplace transforms on p. 317 of the textbook will be provided during the exam.

1. Higher order equations.

   (a) Determine whether these 3 functions are linearly independent:

   $$f_1(t) = 2t - 3, \quad f_2(t) = 2t^2 + 1, \quad f_3(t) = 3t^2 + t \tag{1}$$

   (b) Find the general solution of the sixth-order equation

   $$y^{(6)} - y'' = 0 \tag{2}$$

2. Power series.

   (a) Find the radius of convergence of the series

   $$\sum_{n=0}^{\infty} \frac{n}{2^n} x^n \tag{3}$$

   (b) Re-index this series so that the general term involves x^n:

   $$\sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} + x \sum_{k=1}^{\infty} k a_k x^{k-1} \tag{4}$$

   (c) Seek a power series solution of the following equation at the point $x_0 = 0$:

   $$(4 - x^2) y'' + 2y = 0 \tag{5}$$

   Find the recurrence relation, and determine the first four terms of the solution that begins with $a_0 = 0$, $a_1 = 1$.

   (d) Find the general solution for $x > 0$ of the differential equation:

   $$x^2 y'' - x y' + y = 0 \tag{6}$$

   Note: this equation is an Euler equation with a singular point at $x_0 = 0$.

3. Laplace transform. (A copy of the table on p. 317 of the textbook will be provided during the exam.)

   (a) Write the definition of the Laplace transform of a function $f(t)$.

   (b) Here are some functions of s. Find their inverse Laplace transforms. The answer may have discontinuities or delta functions in it.

   $$F(s) = \frac{2s+1}{s^2 - 2s + 2} \tag{7}$$

   $$F(s) = \frac{2e^{-2s}}{s^2 - 4} \tag{8}$$

   $$F(s) = \frac{e^{-s} + e^{-2s} - 2e^{-3s}}{s} \tag{9}$$

(c) Suppose $F(s)$ is the Laplace transform of $f(t)$. What is the inverse transform of $\frac{F(s)}{s^2+1}$?

(d) For each of following equations, solve for $Y(s) = \mathcal{L}\{y(t)\}$.

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1 \tag{10}$$

$$y'' - 2y' + 2y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 1 \tag{11}$$

$$y'' + 4y = u_\pi(t) - u_{3\pi}(t), \quad y(0) = 0, \quad y'(0) = 0 \tag{12}$$

$$y'' + 2y' + 2y = \delta(t - \pi), \quad y(0) = 1, \quad y'(0) = 0 \tag{13}$$

4. First order systems.

(a) Write this second order equation as a first order system of equations.

$$y'' + 2y' + 3y = 0 \tag{14}$$

(b) In each of these two cases, find the eigenvalues and eigenvectors of the matrix $\mathbf{A}$, and find the general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$

$$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \tag{15}$$

$$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \tag{16}$$