M 427K PRACTICE FOR SECOND MIDTERM EXAM

These problems are representative of the problems that will be on the second midterm exam. This list of problems is *longer* than the actual exam will be. All of these problems are similar to problems that were assigned for homework. A copy of the table of Laplace transforms on p. 317 of the textbook will be provided during the exam.

1. Higher order equations.

   (a) Determine whether these 3 functions are linearly independent:

$$f_1(t) = 2t - 3, \quad f_2(t) = 2t^2 + 1, \quad f_3(t) = 3t^2 + t \tag{1}$$

   Consider a linear relation between the three functions:

$$k_1 f_1 + k_2 f_2 + k_3 f_3 = 0 \tag{2}$$

   Where k_1, k_2, k_3 are constants. If the only such relation is when $k_1 = k_2 = k_3 = 0$, the functions are linearly independent, while if there is any other relation, they are dependent. Using the definitions of f_1, f_2, f_3 and grouping like terms we find

$$k_1(2t - 3) + k_2(2t^2 + 1) + k_3(3t^2 + t) = 0 \tag{3}$$

$$(2k_2 + 3k_3)t^2 + (2k_1 + k_3)t + (-3k_1 + k_2) = 0 \tag{4}$$

   This equation is equivalent to the three equations

$$2k_2 + 3k_3 = 0 \tag{5}$$

$$2k_1 + k_3 = 0 \tag{6}$$

$$-3k_1 + k_2 = 0 \tag{7}$$

   The second and third equations mean $k_3 = -2k_1$ and $k_2 = 3k_1$. Plugging this into the first equation, we get $0k_1 = 0$, so k_1 is unconstrained. In fact $k_1 = 1$, $k_2 = 3$, and $k_3 = -2$ gives a linear relation, so these three functions are not linearly independent.

   (b) Find the general solution of the sixth-order equation

$$y^{(6)} - y'' = 0 \tag{8}$$

   The characteristic equation is $r^6 - r^2 = 0$. This factors as

$$r^6 - r^2 = r^2(r^4 - 1) = r^2(r^2 - 1)(r^2 + 1) = r^2(r - 1)(r + 1)(r^2 + 1) \tag{9}$$

   The roots of the characteristic equation are a double root $r = 0$, single real roots $r = 1$ and $r = -1$, and a complex conjugate pair $r = i$, $r = -i$. The double root at 0 gives solutions $y_1 = 1$ and $y_2 = t$, the single real roots give $y_3 = e^t$ and $y_4 = e^{-t}$, and the complex conjugate pair give $y_5 = \cos(t)$ and $y_6 = \sin(t)$. The general solution is

$$y = c_1 + c_2 t + c_3 e^t + c_4 e^{-t} + c_5 \cos(t) + c_6 \sin(t) \tag{10}$$

2. Power series.

(a) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n}{2^n} x^n \tag{11}$$

Ratio test:

$$L = \lim_{n\to\infty} \frac{|(n+1)2^{-(n+1)}x^{n+1}|}{|n2^{-n}x^n|} = \lim_{n\to\infty} \frac{n+1}{n}\frac{|x|}{2} = \frac{|x|}{2} \tag{12}$$

The series converges when $L < 1$, that is, $|x| < 2$. So the radius of convergence is 2.

(b) Re-index this series so that the general term involves x^n:

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{k=1}^{\infty} k a_k x^{k-1} \tag{13}$$

Shift the first series by $n = m - 2$, and bring the factor of x into the second series to get

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{k=1}^{\infty} k a_k x^k \tag{14}$$

Notice that in the second series, we can start the sum at $k = 0$ without changing the value, since $0a_0x^0 = 0$. Then just rename k to n and group the terms

$$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + na_n]x^n \tag{15}$$

(c) Seek a power series solution of the following equation at the point $x_0 = 0$:

$$(4 - x^2)y'' + 2y = 0 \tag{16}$$

Find the recurrence relation, and determine the first four terms of the solution that begins with $a_0 = 0$, $a_1 = 1$.
We let $y = \sum_{n=0}^{\infty} a_n x^n$. The term $4y''$ becomes $\sum_{n=0}^{\infty} 4(n+2)(n+1)a_{n+2}x^n$ (after a shift by 2 in the index). The term $-x^2y''$ becomes $\sum_{n=0}^{\infty} -n(n-1)a_n x^n$ (no shift necessary). Thus we get

$$\sum_{n=0}^{\infty}[4(n+2)(n+1)a_{n+2} - n(n-1)a_n + 2a_n]x^n = 0 \tag{17}$$

$$4(n+2)(n+1)a_{n+2} - n(n-1)a_n + 2a_n = 0 \tag{18}$$

The recursion relation is

$$a_{n+2} = \frac{n(n-1) - 2}{4(n+2)(n+1)}a_n \tag{19}$$

Starting with $a_0 = 0$, $a_1 = 1$, we get $a_2 = 0$, and

$$a_3 = \frac{1(1-1) - 2}{4(1+2)(1+1)}a_1 = -\frac{1}{12} \tag{20}$$

So the solution is $y = x - x^3/12 + \cdots$.

(d) Find the general solution for $x > 0$ of the differential equation:

$$x^2y'' - xy' + y = 0 \tag{21}$$

Note: this equation is an Euler equation with a singular point at $x_0 = 0$.
We try a solution of the form $y = x^r$. Then we must have

$$r(r-1)x^r - rx^r + x^r = 0 \tag{22}$$

and r must solve

$$0 = r(r-1) - r + 1 = r^2 - 2r + 1 = (r-1)^2 \tag{23}$$

Since $r = 1$ is a double root, the solutions are x^r and $x^r \ln x$, that is x and $x \ln x$. The general solution is

$$y = c_1 x + c_2 x \ln x \tag{24}$$

3. Laplace transform. (A copy of the table on p. 317 of the textbook will be provided during the exam.)

(a) Write the definition of the Laplace transform of a function $f(t)$.

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt \tag{25}$$

(b) Here are some functions of s. Find their inverse Laplace transforms. The answer may have discontinuities or delta functions in it.

$$F(s) = \frac{2s+1}{s^2 - 2s + 2} \tag{26}$$

We complete the square $s^2 - 2s + 2 = (s-1)^2 + 1$. By looking at

$$\mathcal{L}\{e^t \sin t\} = \frac{1}{(s-1)^2 + 1^2} \quad \mathcal{L}\{e^t \cos t\} = \frac{s-1}{(s-1)^2 + 1^2} \tag{27}$$

$$F(s) = 2\left(\frac{s-1}{(s-1)^2 + 1^2}\right) + 3\left(\frac{1}{(s-1)^2 + 1^2}\right) \tag{28}$$

We see $f(t) = 2e^t \cos t + 3e^t \sin t$.

$$F(s) = \frac{2e^{-2s}}{s^2 - 4} \tag{29}$$

The factor of e^{-2s} gives a step function $u_2(t)$, which is multiplied by the inverse transform of $G(s) = \frac{2}{s^2-4}$ shifted by 2. Looking in the table, we see $g(t) = \sinh 2t$, so

$$f(t) = u_2(t)g(t-2) = u_2(t)\sinh 2(t-2) \tag{30}$$

$$F(s) = \frac{e^{-s} + e^{-2s} - 2e^{-3s}}{s} \tag{31}$$

Since e^{-cs}/s is the inverse transform of $u_c(t)$, the answer is

$$g(t) = u_1(t) + u_2(t) - 2u_3(t) \tag{32}$$

(c) Suppose $F(s)$ is the Laplace transform of $f(t)$. What is the inverse transform of $\frac{F(s)}{s^2+1}$? This function is $G(s)F(s)$, where $G(s) = \frac{1}{s^2+1}$. This corresponds to $g(t) = \sin t$. By the convolution formula,

$$\mathcal{L}^{-1}\{G(s)F(s)\} = \int_0^t g(t-\tau)f(\tau)\,d\tau = \int_0^t \sin(t-\tau)f(\tau)\,d\tau \tag{33}$$

(d) For each of following equations, solve for $Y(s) = \mathcal{L}\{y(t)\}$.

$$y'' - 4y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 1 \tag{34}$$

y'' becomes $s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s - 1$. y' becomes $sY(s) - y(0) = sY(s) - 1$. So the equation becomes

$$s^2Y(s) - s - 1 - 4(sY(s) - 1) + 4Y(s) = 0 \tag{35}$$

$$(s^2 - 4s + 4)Y(s) - s + 3 = 0 \tag{36}$$

$$Y(s) = \frac{s-3}{s^2 - 4s + 4} \tag{37}$$

$$y'' - 2y' + 2y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 1 \tag{38}$$

y'' beomes $s^2Y(s) - 1$. y' becomes $sY(s)$. e^{-t} beomes $\frac{1}{s+1}$. The equation becomes

$$s^2Y(s) - 1 - 2(sY(s)) + 2Y(s) = \frac{1}{s+1} \tag{39}$$

$$Y(s) = \frac{1}{s^2 - 2s + 2}\left(1 + \frac{1}{s+1}\right) \tag{40}$$

$$y'' + 4y = u_\pi(t) - u_{3\pi}(t), \quad y(0) = 0, \quad y'(0) = 0 \tag{41}$$

y'' becomes $s^2Y(s)$. $u_\pi(t) - u_{3\pi}(t)$ becomes $(e^{-\pi s} - e^{-3\pi s})/s$. The equation beomes

$$s^2Y(s) + 4Y(s) = (e^{-\pi s} - e^{-3\pi s})/s \tag{42}$$

$$Y(s) = \frac{e^{-\pi s} - e^{-3\pi s}}{s(s^2 + 4)} \tag{43}$$

$$y'' + 2y' + 2y = \delta(t - \pi), \quad y(0) = 1, \quad y'(0) = 0 \tag{44}$$

y'' becomes $s^2Y(s) - s$. y' becomes $sY(s) - 1$. $\delta(t-\pi)$ becomes $e^{-\pi s}$. The equation becomes

$$s^2Y(s) - s + 2(sY(s) - 1) + 2Y(s) = e^{-\pi s} \tag{45}$$

$$Y(s) = \frac{s + 2 + e^{-\pi s}}{s^2 + 2s + 2} \tag{46}$$

4. First order systems.

(a) Write this second order equation as a first order system of equations.

$$y'' + 2y' + 3y = 0 \tag{47}$$

Introduce new variables $x_1 = y$ and $x_2 = y'$. Then the equation becomes the system

$$x_1' = x_2 \tag{48}$$

$$x_2' = -2x_2 - 3x_1 \tag{49}$$

This can also be written

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{50}$$

(b) In each of these two cases, find the eigenvalues and eigenvectors of the matrix $\mathbf{A}$, and find the general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$

$$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \tag{51}$$

The characteristic equation is

$$0 = \det \begin{pmatrix} 3-r & -2 \\ 2 & -2-r \end{pmatrix} = (3-r)(-2-r) + 4 = r^2 - r - 2 = (r-2)(r+1) \tag{52}$$

The eigenvalues are $r_1 = 2$ and $r_2 = -1$.

For $r_1 = 2$, an eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. For $r_2 = -1$, an eigenvector is $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The general solution is

$$\mathbf{x}(t) = c_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{53}$$

$$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \tag{54}$$

The characteristic equation is

$$0 = \det \begin{pmatrix} 3-r & -2 \\ 4 & -1-r \end{pmatrix} = (3-r)(-1-r) + 8 = r^2 - 2r + 5 \tag{55}$$

The roots are

$$r = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i \tag{56}$$

An eigenvector for $r = 1 \pm 2i$ is $\mathbf{v} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$. Thus a complex solution is $e^{(1+2i)t} \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$

Using $e^{(1+2i)t} = e^t(\cos 2t + i \sin 2t)$, we get

$$e^{(1+2i)t} \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} e^t(\cos 2t + i \sin 2t) \\ e^t(\cos 2t + i \sin 2t)(1-i) \end{pmatrix} \tag{57}$$

The real part is

$$\mathbf{u}(t) = \begin{pmatrix} e^t \cos 2t \\ e^t(\cos 2t + \sin 2t) \end{pmatrix} \tag{58}$$

The imaginary part is

$$\mathbf{v}(t) = \begin{pmatrix} e^t \sin 2t \\ e^t(\sin 2t - \cos 2t) \end{pmatrix} \tag{59}$$

The general solution is $\mathbf{x}(t) = c_1 \mathbf{u}(t) + c_2 \mathbf{v}(t)$.