NAME: Solutions

EID:

M 427K Exam 1 Version B **October 9, 2012** **Instructor: James Pascaleff**

Problem	Possible	Actual
1	16	
2	20	
3	12	
4	16	
5	16	
6	12	
7	8	
Total	100	

INSTRUCTIONS:

- Do all work on these sheets.
- Show all work.
- No books, notes, calculators, or other electronic devices.

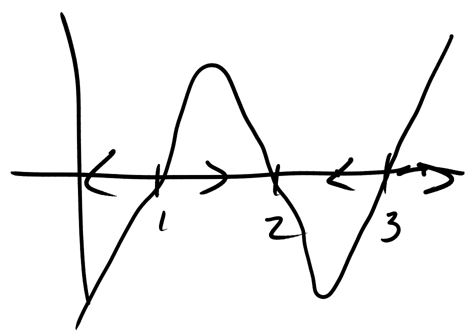
1. Autonomous Equations. (16 points) Consider the first order autonomous equation

$$\frac{dy}{dt} = (y-1)(y-2)(y-3)$$

(a) (4 points) Find the critical points of the equation.

$y=1$, $y=2$, $y=3$

(b) (4 points) Draw a plot of $f(y) = (y-1)(y-2)(y-3)$, and use it to determine where solutions are increasing or decreasing.



$y<1$ decreasing

$1<y<2$ increasing

$2<y<3$ decreasing

$3<y$ increasing

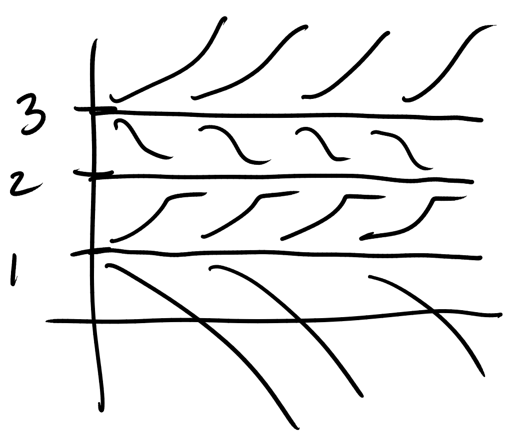
(c) (4 points) Determine whether each equalibrium is stable, unstable, or semistable.

$y=1$ unstable

$y=2$ stable

$y=3$ unstable

(d) (4 points) Make a plot of the solutions to the equation.



2. First Order Solution Techniques. (20 points) For parts (a) and (b), Find the general solution to the equation, either by finding a formula for y, or by finding an implicit equation for y.

(a) (8 points) $\frac{dy}{dx} = yx/(1+y^2)$

$$\frac{1+y^2}{y}\frac{dy}{dx} = x$$

$$\int\left(\frac{1}{y}+y\right)dy = \int x\,dx$$

$$\boxed{\ln|y| + \frac{1}{2}y^2 = \frac{1}{2}x^2 + C}$$ implicit equation

(b) (8 points) $y' + (2/t)y = t^2 + t^{-2}$ (for $t > 0$)

$$u = e^{\int 2/t\,dt} = e^{2\ln t} = t^2$$

$$t^2y' + 2ty = t^4 + 1$$

$$(t^2y)' = t^4 + 1$$

$$t^2y = \int(t^4+1)dt = \frac{1}{5}t^5 + t + C$$

$$y = \frac{1}{5}t^3 + \frac{1}{t} + \frac{C}{t^2}$$

(c) (4 points) Is the equation $(2x + 4xy) + (2x^2)\frac{dy}{dx} = 0$ exact? Explain how you know, but do not solve it.

$$\underbrace{(2x+4xy)}_{M} + \underbrace{(2x^2)}_{N}\frac{dy}{dx} = 0$$

$M_y = 4x \quad N_x = 4x$ equal, so it is exact!

3. Complex Exponentials. (12 points)

(a) (6 points) Use Euler's formula to write $e^{5+\pi i} + e^{(\pi/2)i}$ in the form $a + bi$ where a and b are real.

$$e^{5+\pi i} = e^5 e^{\pi i} = e^5(\cos\pi + i\sin\pi) = e^5(-1+0i)$$
$$= -e^5$$

$$e^{(\pi/2)i} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i\cdot 1 = i$$

$$\text{Total} = \boxed{-e^5 + i}$$

(b) (6 points) Find nonzero constants c_1 and c_2 so that $c_1e^{4it} + c_2e^{-4it}$ is real for all t. There are many correct answers.

$c_1 = c_2 = 1$

$$e^{4it} + e^{-4it} = \cos 4t + i\sin 4t + \cos(-4t) + i\sin(-4t)$$
$$= \cos 4t + i\sin 4t + \cos(4t) - i\sin(4t)$$
$$= 2\cos 4t$$

which is real.

4. Homogeneous Second Order Linear Equations. (16 points) For each of the following equations, find a pair of solutions that form a fundamental set. You do not need to derive your answer from first principles. If it happens the roots are complex, you should find a pair of real solutions (half credit for complex solutions only).

(a) (4 points) $y'' - 3y' + 7y = 0$

$$r = \frac{3 \pm \sqrt{9-28}}{2} = \frac{3 \pm \sqrt{19}\, i}{2}$$

$$y_1 = e^{\frac{3}{2}t} \cos\left(\frac{\sqrt{19}}{2} t\right) \quad y_2 = e^{3/2\, t} \sin\left(\frac{\sqrt{19}}{2} t\right)$$

(b) (4 points) $y'' + 8y' + 16y = 0$

$$r = \frac{-8 \pm \sqrt{64 - 4 \cdot 16}}{2} = \frac{-8}{2} = -4$$

$$y_1 = e^{-4t} \quad y_2 = te^{-4t}$$

(c) (4 points) $y'' - 4y' + 3y = 0$

$$r = \frac{4 \pm \sqrt{16 - 4 \cdot 3}}{2} = \frac{4 \pm 2}{2} = 2 \pm 1 = 1 \text{ or } 3$$

$$y_1 = e^{t} \quad y_2 = e^{3t}$$

(d) (4 points) Find the Wronskian of your solutions to part (c).

$$y_1 y_2' - y_2 y_1' = e^{t}(3e^{3t}) - e^{3t} e^{t}$$
$$= 3e^{4t} - e^{4t} = 2e^{4t}$$

5. NONHOMOGENEOUS EQUATIONS. (16 points) Solve the initial value problem

$$y'' + y' - 2y = 10\cos(t), \quad y(0) = -3, \quad y'(0) = 1$$

You may use any methods, but you must show your work.

$y = A\cos t + B\sin t$

$y' = -A\sin t + B\cos t$

$y'' = -A\cos t - B\sin t$

$y'' + y' - 2y$

$= -A\cos t - B\sin t$

$+ B\cos t - A\sin t$

$-2A\cos t - 2B\sin t$

$= (-A + B - 2A)\cos t + (-B - A - 2B)\sin t$

$= (-3A + B)\cos t + (-3B - A)\sin t$

so need

$-3A + B = 10$

$-3B - A = 0$

$A = -3B$

$-3(-3B) + B = 10$

$10B = 10 \Rightarrow B = 1 \Rightarrow A = -3$ so $y = -3\cos t + \sin t$

roots $r = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} = -2$ and 1

$y = c_1 e^{t} + c_2 e^{-2t} - 3\cos t + \sin t$ $\quad y(0) = c_1 + c_2 - 3$

$y' = c_1 e^{t} - 2c_2 e^{-2t} + 3\sin t + \cos t$ $\quad y'(0) = c_1 - 2c_2 + 1$

$c_1 + c_2 - 3 = -3$
$c_1 - 2c_2 + 1 = 1$
$\Rightarrow$
$c_1 + c_2 = 0$
$c_1 - 2c_2 = 0$
$\Rightarrow c_1 = c_2 = 0$

so $y = -3\cos t + \sin t$

6. Theory of Linear Equations. (12 points) Consider the second order linear differential operator

$$L[y] = y'' + \sin(t)y' + e^{2t}y$$

Suppose we want to solve the nonhomogeneous equation

$$L[y] = 4\frac{\sin(t)}{t} + 3\ln(t)$$

This is very difficult to do from scratch, but suppose that a very smart colleague finds for us functions Y_1 and Y_2 solving the equations

$$L[Y_1] = \frac{\sin(t)}{t}, \qquad L[Y_2] = 2\ln(t)$$

How can we use Y_1 and Y_2 to get a solution to the original equation?

Superposition principle

$$Y = 4Y_1 + \frac{3}{2}Y_2$$

$$L[Y] = L[4Y_1 + \frac{3}{2}Y_2]$$

$$= 4L[Y_1] + \frac{3}{2}L[Y_2]$$

$$= 4\frac{\sin(t)}{t} + 3\ln(t)$$

7. VIBRATIONS. (8 points) Suppose that a damped mass-on-a-spring is described by the equation

$$5u'' + \gamma u' + 10u = 0$$

When γ is small, the solutions oscillate, while when γ is large they do not. Find the value of γ where the cross-over between these two behaviors occurs. In other words, find the value of γ for which the system is critically damped.

Discriminant $= 0$

$$0 = \gamma^2 - 4km = \gamma^2 - 4 \cdot 10 \cdot 5 = \gamma^2 - 200$$

$$\gamma^2 = 200$$

$$\gamma = \sqrt{200}$$