NAME: Solutions

EID:

M 427K Exam 1 Version A **October 9, 2012** **Instructor: James Pascaleff**

Problem	Possible	Actual
1	16	
2	20	
3	12	
4	16	
5	16	
6	12	
7	8	
Total	100	

INSTRUCTIONS:

- Do all work on these sheets.
- Show all work.
- No books, notes, calculators, or other electronic devices.

1. Autonomous Equations. (16 points) Consider the first order autonomous equation

$$\frac{dy}{dt} = -y(y-1)(y-2)$$

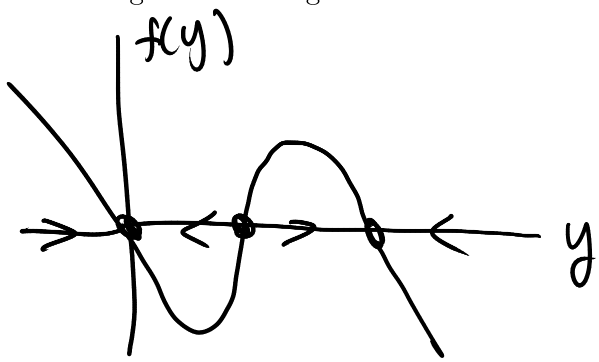
(a) (4 points) Find the critical points of the equation.

$y=0$

$y=1$

$y=2$

(b) (4 points) Draw a plot of $f(y) = -y(y-1)(y-2)$, and use it to determine where solutions are increasing or decreasing.



$y<0$ increasing

$0<y<1$ decreasing

$1<y<2$ increasing

$2<y$ decreasing

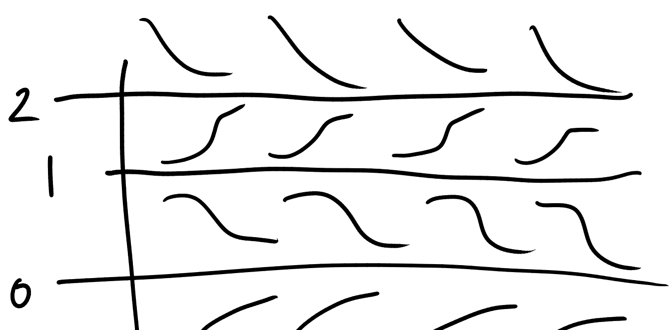
(c) (4 points) Determine whether each equalibrium is stable, unstable, or semistable.

$y=0$ stable

$y=1$ unstable

$y=2$ stable

(d) (4 points) Make a plot of the solutions to the equation.



2. First Order Solution Techniques. (20 points) For parts (a) and (b), Find the general solution to the equation, either by finding a formula for y, or by finding an implicit equation for y.

(a) (8 points) $y' + (1/t)y = t^3 + t^{-1}$ (for $t > 0$)

$\mu = e^{\int \frac{1}{t} dt} = e^{\ln t} = t$

$ty' + y = t^4 + 1$

$(ty)' = t^4 + 1$

$ty = \int (t^4 + 1) dt = \frac{1}{5} t^5 + t + c$

$y = \frac{1}{5} t^4 + 1 + \frac{c}{t}$

(b) (8 points) $\frac{dy}{dx} = e^{x-y}$

$e^y \frac{dy}{dx} = e^x$

$\int e^y \frac{dy}{dx} dx = \int e^x dx$

$e^y = e^x + C$

$y = \ln(e^x + c)$

(c) (4 points) Is the equation $(2x + 4y) + (2y^2 x)\frac{dy}{dx} = 0$ exact? Explain how you know, but do not solve it.

$\underbrace{(2x+4y)}_{= M} \quad \underbrace{(2y^2x)}_{= N}$

$M_y = 4 \qquad N_x = 2y^2$ not equal, so no, it isn't exact.

3. Complex Exponentials. (12 points)

(a) (6 points) Use Euler's formula to write $e^{1+(\pi/2)i} + e^{\pi i}$ in the form $a + bi$ where a and b are real.

$$e^{1+(\pi/2)i} = e^1 \cdot e^{(\pi/2)i} = e\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= e(0 + i \cdot 1) = ei$$

$$e^{\pi i} = \cos\pi + i\sin\pi = -1 + 0i = -1$$

so total $= \boxed{-1 + ei}$

(b) (6 points) Find nonzero constants c_1 and c_2 so that $c_1 e^{2it} + c_2 e^{-2it}$ is real for all t. There are many correct answers.

$c_1 = c_2 = 1$:

$$e^{2it} + e^{-2it} = \cos(2t) + i\sin(2t) + \cos(-2t) + i\sin(-2t)$$

$$= \cos(2t) + i\sin(2t) + \cos(2t) - i\sin(2t)$$

$$= 2\cos(2t)$$ which is real.

4. Homogeneous Second Order Linear Equations. (16 points) For each of the following equations, find a pair of solutions that form a fundamental set. You do not need to derive your answer from first principles. If it happens the roots are complex, you should find a pair of real solutions (half credit for complex solutions only).

(a) (4 points) $y'' + 6y' + 9y = 0$

$$r = \frac{-6 \pm \sqrt{36-36}}{2} = \frac{-6}{2} = -3$$

$$y_1 = e^{-3t} \qquad y_2 = t e^{-3t}$$

(b) (4 points) $y'' - 3y' + 6y = 0$

$$r = \frac{+3 \pm \sqrt{9-4\cdot 6}}{2} = \frac{3 \pm \sqrt{15}\, i}{2}$$

$$y_1 = e^{\frac{3}{2}t} \cos\left(\frac{\sqrt{15}}{2}t\right) \qquad y_2 = e^{\frac{3}{2}t} \sin\left(\frac{\sqrt{15}}{2}t\right)$$

(c) (4 points) $y'' - 3y' + 2y = 0$

$$r = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = 1 \text{ or } 2$$

$$y_1 = e^{t} \qquad y_2 = e^{2t}$$

(d) (4 points) Find the Wronskian of your solutions to part (c).

$$y_1 y_2' - y_2 y_1' = e^{t}(2e^{2t}) - e^{2t} e^{t}$$
$$= 2e^{3t} - e^{3t} = e^{3t}$$

5. Nonhomogeneous Equations. (16 points) Solve the initial value problem

$$y'' + y' - 2y = 4t^2, \quad y(0) = -3, \quad y'(0) = 1$$

You may use any methods, but you must show your work.

$y = At^2 + Bt + C$ $\quad y'' + y' - 2y = 2A + 2At + B - 2At^2 - 2Bt - 2C$

$y' = 2At + B$

$y'' = 2A$ $\quad = (-2A)t^2 + (2A - 2B)t + 2A + B - 2C$

Need $-2A = 4 \Rightarrow A = -2$

$2A - 2B = 0 \Rightarrow B = A = -2$

$2A + B - 2C = 0 \Rightarrow -4 - 2 - 2C = 0 \quad -6 = 2C \quad C = -3$

$y = -2t^2 - 2t - 3$

roots $\dfrac{-1 \pm \sqrt{1+8}}{2} = \dfrac{-1 \pm 3}{2} = -2$ or 1

$y = c_1 e^t + c_2 e^{-2t} - 2t^2 - 2t - 3 \quad y(0) = c_1 + c_2 - 3$

$y' = c_1 e^t - 2c_2 e^{-2t} - 4t - 2 \quad y'(0) = c_1 - 2c_2 - 2$

$c_1 + c_2 - 3 = -3 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$c_1 - 2c_2 - 2 = 1 \Rightarrow c_1 - 2c_2 = 3 \Rightarrow 3c_1 = 3 \quad c_1 = 1$

so $\boxed{y = e^t - e^{-2t} - 2t^2 - 2t - 3}$

6. Theory of Linear Equations. (12 points) Consider the second order linear differential operator

$$L[y] = y'' + 5e^t y' + \cos(t) y$$

Suppose we want to solve the nonhomogeneous equation

$$L[y] = \pi \arctan(t) + 5\ln(t)$$

This is very difficult to do from scratch, but suppose that a very smart colleague finds for us functions Y_1 and Y_2 solving the equations

$$L[Y_1] = \arctan(t), \qquad L[Y_2] = 2\ln(t)$$

How can we use Y_1 and Y_2 to get a solution to the original equation?

Superposition principle

$$Y = \pi Y_1 + \frac{5}{2} Y_2$$

$$L[Y] = L[\pi Y_1 + \frac{5}{2} Y_2]$$

$$= \pi L[Y_1] + \frac{5}{2} L[Y_2]$$

$$= \pi \arctan(t) + 5\ln(t)$$

7. VIBRATIONS. (8 points) Suppose that a damped mass-on-a-spring is described by the equation

$$10u'' + \gamma u' + 4u = 0$$

When γ is small, the solutions oscillate, while when γ is large they do not. Find the value of γ where the cross-over between these two behaviors occurs. In other words, find the value of γ for which the system is critically damped.

Discriminant = 0

$0 = \gamma^2 - 4km = \gamma^2 - 4 \cdot 4 \cdot 10 = \gamma^2 - 160$

$\gamma^2 = 160$

$\gamma = \sqrt{160}$