Solutions 9

5.2 System functions for amount of time X

PDF is $f(x) = \begin{cases} Cxe^{-x/2} & x>0 \\ 0 & x \le 0 \end{cases}$

x is in months

what is $P(X \ge 5)$?

First find $\int xe^{-x/2}dx = -2xe^{-x/2} - \int (-2)e^{-x/2}dx$

(integration by parts)

$= -2xe^{-x/2} - 4e^{-x/2}$ + constant

So $1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} Cxe^{-x/2}dx = C\left[-2xe^{-x/2} - 4e^{-x/2}\right]_0^{\infty}$

$= C\left[0 - \left(-2\cdot 0\cdot e^{-0/2} - 4e^{-0/2}\right)\right] = C\cdot 4$

so $C = \frac{1}{4}$

$P(X \ge 5) = \int_5^{\infty} f(x)dx = \int_5^{\infty} \frac{1}{4}xe^{-x/2}dx$

$= \frac{1}{4}\left[-2xe^{-x/2} - 4e^{-x/2}\right]_5^{\infty}$

$= \frac{1}{4}\left[0 - \left(-2\cdot 5\cdot e^{-5/2} - 4e^{-5/2}\right)\right] = \frac{1}{4}(14)e^{-5/2} = \frac{7}{2}e^{-5/2}$

5.3 Consider $f(x) = \begin{cases} C(2x-x^3) & 0<x<\frac{5}{2} \\ 0 & \text{otherwise} \end{cases}$

$f(2) = C(2\cdot 2 - 2^3) = C(4-8) = C(-4)$

$f(1) = C(2\cdot 1 - 1^3) = C(2-1) = C\cdot 1$

Since the probability density function of a random variable must be nonnegative,

$0 \le f(2) = C(-4) \Rightarrow C \le 0$
$0 \le f(1) = C(1) \Rightarrow C \ge 0$
So $C=0$ is the only possibility.

But then $f(x)=0$ everywhere, and $\int_{-\infty}^{\infty} f(x)dx = 0$, which is impossible if f is a PDF

for $f(x) = \begin{cases} C(2x-x^2) & 0<x<\frac{5}{2} \\ 0 & \text{otherwise} \end{cases}$

we see $f(1) = C(2-1) = C(1) \Rightarrow C \ge 0$
$f(2.1) = C(4.2 - 4.41) = C(-.21) \Rightarrow C \le 0$

so $C=0$ but $\int_{-\infty}^{\infty} f(x)\,dx$ can't be zero

So neither function can be the PDF of a RV.

5.4 X = life time of device (in hours)

has PDF $f(x) = \begin{cases} \frac{10}{x^2} & x > 10 \\ 0 & x \le 10 \end{cases}$

First compute $\int \frac{10}{x^2}\,dx = -\frac{10}{X} + \text{constant}$

(a) $P\{X > 20\} = \int_{20}^{\infty} \frac{10}{x^2}\,dx = \left[-\frac{10}{x}\right]_{20}^{\infty} = 0 - \left(-\frac{10}{20}\right) = \frac{1}{2}$

(b) Cumulative distribution function $F(a) = \int_{-\infty}^{a} f(x)\,dx$

$$\int_{-\infty}^{a} f(x)\,dx = \int_{10}^{a} \frac{10}{x^2}\,dx = \left[-\frac{10}{x}\right]_{10}^{a} = -\frac{10}{a} - \left(-\frac{10}{10}\right)$$

$$F(a) = 1 - \frac{10}{a}$$

(c) Given six such devices, what is probability at least 3 function for at least 15 hours?

Assume each of 6 devices functions/fails independently
Then we have Bernoulli trials with $n = 6$
and $p = P\{X > 15\}$:

$$p = P\{X \ge 15\} = 1 - P\{X \le 15\} = 1 - F(15) = \frac{10}{15} = \frac{2}{3}$$

$$P(\text{exactly } k \text{ of } 6 \text{ last } 15 \text{ hours}) = \binom{6}{k} p^k (1-p)^{6-k}$$

$$= \binom{6}{k}\left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{6-k} = \binom{6}{k}\frac{2^k}{3^6}$$

$P(\geq 3 \text{ last at least 1 '5'}) = P(3) + P(4) + P(5) + P(6)$

$$= \binom{6}{3}\frac{2^3}{3^6} + \binom{6}{4}\frac{2^4}{3^6} + \binom{6}{5}\frac{2^5}{3^6} + \binom{6}{6}\frac{2^6}{3^6}$$

5.6 Compute $E[X]$:

(a) $f(x) = \begin{cases} \frac{1}{4} x e^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} \frac{1}{4} x^2 e^{-x/2}\,dx$$

integrate by parts $\quad u = x^2 \quad dv = e^{-x/2}dx$

$du = 2x\,dx \quad v = -2e^{-x/2}$

$$\int u\,dv = uv - \int v\,du$$

$$\int \frac{1}{4}x^2 e^{-x/2} dx = \frac{1}{4}x^2(-2e^{-x/2}) - \int \frac{1}{4} 2x(-2) e^{-x/2} dx$$

$$= -\frac{1}{2}x^2 e^{-x/2} + \int x e^{-x/2} dx$$

integrate by parts

$\alpha = x \quad d\beta = e^{-x/2}dx$

$d\alpha = dx \quad \beta = -2e^{-x/2}$

$$= -\frac{1}{2}x^2 e^{-x/2} + (-2xe^{-x/2}) - \int -2e^{-x/2} dx$$

$$= -\frac{1}{2}x^2 e^{-x/2} - 2xe^{-x/2} - 4e^{-x/2} + \text{constant.}$$

$$= \left(-\frac{1}{2}x^2 - 2x - 4\right) e^{-x/2}$$

Check $\frac{d}{dx}\left(-\frac{1}{2}x^2-2x-4\right)e^{-x/2}$

$$= (-x-2)\,e^{-x/2} + \left(-\tfrac{1}{2}x^2-2x-4\right)\left(-\tfrac{1}{2}\right)e^{-x/2}$$

$$= \left(-x-2+\tfrac{1}{4}x^2+x+2\right)e^{-x/2} = \tfrac{1}{4}x^2 e^{-x/2} \quad \checkmark$$

$$E[X] = \int_0^\infty \tfrac{1}{4}x^2 e^{-x/2}\,dx = \left[\left(-\tfrac{1}{2}x^2-2x-4\right)e^{-x/2}\right]_0^\infty$$

Now $\lim\limits_{x\to\infty} x^n e^{-ax} = 0$ for any $n \geq 0$ & $a > 0$

(this may be proved using repeated application of L'Hospital's rule)

Thus $\lim\limits_{x\to\infty}\left(-\frac{1}{2}x^2-2x-4\right)e^{-x/2} = 0$

$$E[X] = 0 - \left(-\tfrac{1}{2}(0)^2-2(0)-4\right)e^{-0/2} = 4$$

(b) $f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

Recall from 5.1 that $c = \frac{3}{4}$

So $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-1}^{1} x\left(\frac{3}{4}\right)(1-x^2)\,dx$

$$= \frac{3}{4}\int_{-1}^{1}(x-x^3)\,dx = \frac{3}{4}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_{-1}^{1}$$

$$= \frac{3}{4}\left[\left(\frac{1}{2}-\frac{1}{4}\right)-\left(\frac{(-1)^2}{2}-\frac{(-1)^4}{4}\right)\right] = \frac{3}{4}\left[\left(\frac{1}{2}-\frac{1}{4}\right)-\left(\frac{1}{2}-\frac{1}{4}\right)\right] = 0$$

(c) $f(x) = \begin{cases} \frac{5}{x^2} & x > 5 \\ 0 & x \le 5 \end{cases}$

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_5^{\infty} x\left(\frac{5}{x^2}\right) dx = \int_5^{\infty} \frac{5}{x} dx$$

$$= \left[5 \ln(x) \right]_5^{\infty} = 5\left(\lim_{x\to\infty} \ln(x) - \ln(5)\right)$$

Now $\lim_{x\to\infty} \ln(x) = \infty$ so this integral diverges

$E[X] = \infty$.

5.7 $f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ and $E[X] = \frac{3}{5}$

Thus we have two unknowns: a, b
and two equations:

$$\int_{-\infty}^{\infty} f(x) dx = 1 \qquad \int_{-\infty}^{\infty} x f(x) dx = E[X] = \frac{3}{5}$$

$$1 = \int_0^1 (a + bx^2) dx = \left[ax + b\frac{x^3}{3} \right]_0^1 = a + \frac{b}{3}$$

$$\frac{3}{5} = \int_0^1 x(a + bx^2) dx = \int_0^1 (ax + bx^3) dx = \left[a\frac{x^2}{2} + b\frac{x^4}{4} \right]_0^1 = \frac{a}{2} + \frac{b}{4}$$

so $\left.\begin{cases} a + \frac{b}{3} = 1 \Rightarrow 3a + b = 3 \\ \frac{a}{2} + \frac{b}{4} = \frac{3}{5} \Rightarrow 2a + b = \frac{12}{5} \end{cases}\right\}$ subtract $\Rightarrow a = 3 - \frac{12}{5}$

$a = \frac{15}{5} - \frac{12}{5} = \frac{3}{5}$

So $b = \frac{12}{5} - 2a = \frac{12}{5} - \frac{6}{5} = \frac{6}{5}$

So $a = \frac{3}{5}$ $b = \frac{6}{5}$

$$f(x) = \begin{cases} \frac{3}{5} + \frac{6}{5}x^2 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

5.8 X = lifetime in hours of device. PDF is:

$f(x) = xe^{-x}$ $x \geq 0$

Expected lifetime $= E[X] = \int_0^\infty x f(x)dx = \int_0^\infty x^2 e^{-x} dx$

[This is similar to 5.6(a)]

$$\int x^2 e^{-x} dx = -x^2 e^{-x} - \int 2x(-e^{-x})dx$$

$$= -x^2 e^{-x} + 2\int x e^{-x} dx$$

$$= -x^2 e^{-x} + 2\left[-xe^{-x} - \int -e^{-x} dx\right]$$

$$= -x^2 e^{-x} - 2xe^{-x} + 2\int e^{-x} dx$$

$$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} = -(x^2+2x+2)e^{-x}$$

Check: $\frac{d}{dx}\left[-(x^2+2x+2)e^{-x}\right] =$

$$= -(2x+2)e^{-x} - (x^2+2x+2)(-e^{-x}) = x^2 e^{-x} \quad ✓$$

So $E[X] = \int_0^\infty x^2 e^{-x} dx = \left[-(x^2+2x+2)e^{-x}\right]_0^\infty$

$$= \left[\lim_{x\to\infty} -(x^2+2x+2)e^{-x}\right] + 2 = 0+2 = 2$$

(since $\lim_{x\to\infty} x^n e^{-x} = 0$ for $n \geq 0$)