# Solutions 8

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4.63 People enter casino at 1 per 2 minutes

Poisson process with rate parameter $\lambda = .5$ per min.

Between 12:00 and 12:05 = 5 min interval

$X =$ Number who enter is Poisson distributed $\lambda t = (.5) \cdot 5 = 2.5$

(a) $P\{X=0\} = e^{-2.5}\dfrac{(2.5)^0}{0!} = e^{-2.5}$

(b) $P\{X \geq 4\} = 1 - P\{X<4\} = 1 - P\{X=0\} - P\{X=1\} - P\{X=2\} - P\{X=3\}$

$= 1 - e^{-2.5} - 2.5\,e^{-2.5} - \dfrac{(2.5)^2}{2}e^{-2.5} - \dfrac{(2.5)^3}{6}e^{-2.5}$

4.70 Coin starts on heads. It is flipped $N(t)$ times

where $N(t)$ is a poisson process with rate $\lambda$

$$P\{N(t)=k\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$$

What is $P\{$coin is showing heads at time $t\}$

(Note: coin is not necessarily fair : $P(\text{heads}) = p$)

$P(\text{coin showing heads at time } t)$

$= P(\text{coin showing heads at time } t \mid N(t)=0)\, P(N(t)=0)$

$+ P(\text{coin showing heads at time } t \mid N(t)>0)\, P(N(t)>0)$

Since coin starts on heads, $N(t)=0 \Rightarrow$ no flips
$\Rightarrow$ coin is showing heads at time $t$

$P(\text{head at time } t \mid N(t)=0) = 1$

If $N(t)>0$, then the coin has been flipped, but all that matters is the <u>last</u> flip, which has probability $p$ of being heads

$P(\text{heads at time } t \mid N(t)>0) = p.$

Also $P(N(t)=0) = e^{-\lambda t}$ $\quad P(N(t)>0) = 1 - e^{-\lambda t}$

so $P(\text{heads at time } t) = 1 \cdot e^{-\lambda t} + p(1-e^{-\lambda t})$

$$= p + (1-p)e^{-\lambda t}$$

4.71 Roulette w/ 38 spaces. Smith bets on 1–12
has $p = \frac{12}{38}$ chance of winning each time

We have here Bernoulli trials with $p = \frac{12}{38}$

(a) P(Smith loses first 5 bets) = P(5 failures in a row)

$$= (1-p)^5 = \left(1 - \frac{12}{38}\right)^5 = \left(\frac{26}{38}\right)^5$$

(b) P(first win occurs on 4th bet) $= (1-p)^3 p$

$$= \left(1-\frac{12}{38}\right)^3\left(\frac{12}{38}\right) = \left(\frac{26}{38}\right)^3\left(\frac{12}{38}\right)$$ (geometric probability distribution)

4.75 A fair coin $\left(p=\frac{1}{2}\right)$ is flipped until heads occurs for 10 th time $X = \#$ of tails

Let $Y$ = total # of flips. Since we get 10 heads and $X$ tails,

$$Y = X + 10$$

Also $Y$ is negative binomial RV w/ $r=10$, $p=\frac{1}{2}$

$$P\{X=k\} = P\{Y=k+10\} = \binom{k+10-1}{10-1}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{k+10-10}$$

$$= \binom{k+9}{9}\left(\frac{1}{2}\right)^{k+10}$$

4.78 Urn w/ 4 white and 4 black. Choose 4 balls if get 2 white, 2 black, stop. otherwise try again

$$P(2\text{ white}/2\text{ black}) = \frac{\binom{4}{2}\binom{4}{2}}{\binom{8}{4}} = \frac{6 \cdot \overset{3}{\cancel{6}}}{\frac{8\cdot\overset{2}{\cancel{7}}\cdot 6\cdot 5}{4\cdot 3\cdot 2}} = \frac{18}{35}$$ (hypergeometric)

Now we're doing Bernoulli trials w/ $p = \frac{18}{35}$

$X$ = # selections to first success (2 white/2 black) is geometric

$$P\{X=n\} = (1-p)^{n-1} p = \left(1-\frac{18}{35}\right)^{n-1}\left(\frac{18}{35}\right) = \left(\frac{17}{35}\right)^{n-1}\left(\frac{18}{35}\right)$$

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4.27 $X$ = geometric w/ parameter $p$.

Show $P\{X=n+k \mid X>n\} = P\{X=k\}$

$$P\{X=n+k\} = (1-p)^{n+k-1} p$$

$$P\{X>n\} = P\{\text{first } n \text{ trials are failures}\} = (1-p)^n$$

$$P\{X=n+k \mid X>n\} = \frac{P\{X=n+k\}}{P\{X>n\}} = \frac{(1-p)^{n+k-1} p}{(1-p)^n}$$

$$= (1-p)^{k-1} p$$

But $P\{X=k\} = (1-p)^{k-1} p$ by PMF of geometric RV.

Conceptual argument

$X>n$ means the first $n$ trials are failures. If we are looking for the first success, we can forget about the first $n$ trials and pretend we are starting anew. Then the probability of getting 1st success on $(n+k)$th trial (after a string of $n$ failures) is like getting 1st success on $k$ th trial, starting from the beginning.

so $P\{X = n+k \mid X > n\} = P\{X = k\}$

4.29 For hypergeometric R.V., compute

$$P\{X=k+1\} / P\{X=k\}$$

$$P\{X=k\} = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}} \quad \text{hypergeometric PMF.}$$

$$\frac{P\{X=k+1\}}{P\{X=k\}} = \frac{\binom{m}{k+1}\binom{N-m}{n-k-1}}{\binom{N}{n}} \Big/ \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$$

$$= \frac{\binom{m}{k+1}\binom{N-m}{n-k-1}}{\cancel{\binom{N}{n}}} \cdot \frac{\cancel{\binom{N}{n}}}{\binom{m}{k}\binom{N-m}{n-k}}$$

$$= \frac{\binom{m}{k+1}}{\binom{m}{k}} \cdot \frac{\binom{N-m}{n-k-1}}{\binom{N-m}{n-k}}$$

**Lemma:** $\displaystyle \binom{a}{b+1} \Big/ \binom{a}{b} = \frac{a-b}{b+1}$

**Proof** $$\frac{\binom{a}{b+1}}{\binom{a}{b}} = \frac{\frac{a!}{(a-b-1)!(b+1)!}}{\frac{a!}{(a-b)!\,b!}} = \frac{\cancel{a!}}{(a-b-1)!\,(b+1)!} \cdot \frac{(a-b)!\,b!}{\cancel{a!}}$$

$$= \frac{(a-b)!}{(a-b-1)!} \cdot \frac{b!}{(b+1)!} = a-b \cdot \frac{1}{b+1} = \frac{a-b}{b+1} \quad \blacksquare$$

Now $$\frac{\binom{m}{k+1}}{\binom{m}{k}} = \frac{m-k}{k+1} \quad (\text{use } a=m,\ b=k)$$

and $$\frac{\binom{N-m}{n-k}}{\binom{N-m}{n-k-1}} = \frac{N-m-n+k+1}{n-k} \quad \left(\text{use } \begin{array}{l} a=N-m \\ b=n-k-1 \end{array}\right)$$

so $$\frac{P\{X=k+1\}}{P\{X=k\}} = \frac{m-k}{k+1} \cdot \frac{n-k}{N-m-n+k+1}$$

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5.1 X continuous RV PDF $f(x)$

$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) value of c?

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-1}^{1} c(1-x^2)\,dx = c\left[x - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= c\left[1 - \frac{1}{3} - \left(-1 - \frac{-1}{3}\right)\right] = c\left[\frac{2}{3} - \left(-\frac{2}{3}\right)\right] = c\,\frac{4}{3}$$

so $c = \frac{3}{4}$

Cumulative distribution function $F(a) = \int_{-\infty}^{a} f(x)\,dx$

If $a \le -1$ $F(a) = \int_{-\infty}^{a} 0\,dx = 0$

If $-1 < a < 1$ $F(a) = \int_{-\infty}^{a} f(x)\,dx = \int_{-1}^{a} \frac{3}{4}(1-x^2)\,dx$

$$= \left[\frac{3}{4}\left(x - \frac{x^3}{3}\right)\right]_{-1}^{a} = \frac{3}{4}\left(a - \frac{a^3}{3}\right) - \frac{3}{4}\underbrace{\left(-1 - \frac{-1}{3}\right)}_{-\frac{2}{3}}$$

$$= \frac{3}{4}\left(a - \frac{a^3}{3}\right) + \frac{1}{2}$$

If $1 \le a$ $F(a) = \int_{-\infty}^{a} f(x)\,dx = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} \frac{3}{4}(1-x^2)\,dx + \int_{1}^{a} 0\,dx = 1$

So $F(a) = \begin{cases} 0 & \text{if } a \leq -1 \\ \frac{3}{4}\left(a - \frac{a^3}{3}\right) + \frac{1}{2} & \text{if } -1 < a < 1 \\ 1 & \text{if } 1 \leq a \end{cases}$