Solutions 7

4.30 A coin is flipped until a tail appears. Let Y denote the number of flips required. Then Y is a random variable and

$$P(Y=n) = P(\underbrace{HH\cdots H}_{n-1}T) = \left(\frac{1}{2}\right)^n$$

The payout in the St. Petersburg lottery is $X = 2^Y$

So $P(X=2^n) = P(Y=n) = \left(\frac{1}{2}\right)^n$

Thus $E[X] = \sum_{n=1}^{\infty} 2^n \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} 1 = \infty$

(a) No. Let us compute

$$P(X \geq 2^k) = P(Y \geq k) = \sum_{n=k}^{\infty} \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^k \frac{1}{1-\frac{1}{2}} = \left(\frac{1}{2}\right)^{k-1}$$

(geometric series)

In order to win at least 10^6, we would need to take $k = \lceil(\log_2 10^6)\rceil = \lceil 6 \log_2 10 \rceil = 20$

$$P(Y \geq 20) = \left(\frac{1}{2}\right)^{19} = \frac{1}{524288}$$

So it is extremely unlikely that we will make back our investment of 10^6 in one try

(b) (SOFT ANSWER) Yes you should play if you can play as much as you want and only have to settle up at the end.

Since $E[X] = \infty > 10^6$ we see that, Though on most games you will not win as much as you spent to play, there will be rare but extremely large wins which set to offset the losses

4.33 Buy papers at 10¢, sell at 15¢

Daily demand is binomial RV with parameters $n=10, p=\frac{1}{3}$

How many papers should he by to maximize his profit.

Let X be a random variable representing demand

so $P(X=i) = \binom{10}{i}\left(\frac{1}{3}\right)^i\left(\frac{2}{3}\right)^{n-i}$ $\quad i=0,1,\ldots,10$

If we buy k papers, we spend $10k$ (in ¢) and then we are able to sell up to k papers

So we actually sell $\min(X,k)$

and we make $15\min(X,k)$

so net profit is $15\min(X,k) - 10k$

Expected profit is $E\left[15\min(X,k) - 10k\right]$

$$= 15\, E[\min(X,k)] - 10k$$

$$E[\min(X,k)] = \sum_{i=0}^{k-1} i \binom{10}{i} \left(\tfrac{1}{3}\right)^i \left(\tfrac{2}{3}\right)^{n-i} + k \sum_{i=k}^{10} \binom{10}{i} \left(\tfrac{1}{3}\right)^i \left(\tfrac{2}{3}\right)^{n-i}$$

Using a computer to tabulate values

k	$15\, E[\min(X,k)] - 10k$
1	4.74
2	8.18
3	8.69 ←
4	5.30
5	-1.5
6	-10.35
7	-20.06
8	-30.01
9	-40.00
10	-50.00

So the expected profit is maximized when we buy 3 papers.

4.38 Suppose $E[X] = 1$ and $Var[X] = 5$

First find $E[X^2] = Var[X] + (E[X])^2 = 5 + (1)^2 = 6$

$\left(\text{using } Var[X] = E[X^2] - (E[X])^2\right)$

So: $E[(2+X)^2] = E[4 + 4X + X^2]$

$= 4 + 4E[X] + E[X^2] = 4 + 4(1) + (6) = 14$

$$Var[4+3X] = E[(4+3X)^2] - (E[4+3X])^2$$

$$E[4+3X] = 4+3E[X] = 4+3(1) = 7$$

$$E[(4+3X)^2] = E[16+24X+9X^2]$$

$$= 16+24E[X]+9E[X^2] = 16+24(1)+9(6) = 94$$

$$Var[4+3X] = 94-(7)^2 = 94-49 = 45$$

4.41 Man claims to have ESP. fair coin is flipped 10 times
Man gets 7 out of 10 correct. What is probability of doing at least this well with random guessing

Random guessing has probability of success $p=\frac{1}{2}$ on each flip. So # of correct guesses is a Bernoulli Random variable with $n=10$, $p=\frac{1}{2}$

$$P(X=7) = \binom{10}{7}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3 = \frac{10!}{3!\,7!}\frac{1}{1024} = \frac{10\cdot 9\cdot 8}{3\cdot 2}\frac{1}{1024}$$

$$= \frac{10\cdot 3\cdot 4}{1024} = \frac{120}{1024} \approx .117$$

$$P(X=8) = \binom{10}{8}\frac{1}{1024} = \frac{10\cdot 9}{2}\cdot\frac{1}{1024} = \frac{45}{1024} \approx 0.044$$

$$P(X=9) = \binom{10}{9}\frac{1}{1024} = \frac{10}{1024} \approx 0.010$$

$$P(X=10) = \binom{10}{10}\frac{1}{1024} = \frac{1}{1024} \approx 0.001$$

$P(X\geq 7) = \frac{176}{1024} \approx .172 > \frac{1}{6}$ which isn't particularly small.

4.57 Suppose # of accidents on a highway each day is a Poisson Random Variable with $\lambda = 3$

(a) $P(X \geq 3) = 1 - P(X=0) - P(X=1) - P(X=2)$

$$= 1 - e^{-3} - 3e^{-3} - \frac{(3)^2}{2} e^{-3} = 1 - 8.5e^{-3} \approx 0.577$$

(b) $$P(X \geq 3 \mid X \geq 1) = \frac{P(X \geq 3 \text{ and } X \geq 1)}{P(X \geq 1)} = \frac{P(X \geq 3)}{P(X \geq 1)}$$

$P(X \geq 1) = 1 - P(X=0) = 1 - e^{-3} \approx 0.950$

so $P(X \geq 3 \mid X \geq 1) = 0.607$

4.59 play lottery 50 times $p = \frac{1}{100}$ in each

X = # of wins is binomial with parameters $n = 50$, $p = \frac{1}{100}$

We will approximate X by a Poisson RV. with $\lambda = np = \frac{1}{2}$

(a) $P(X \geq 1) = 1 - P(X=0) = 1 - e^{-\frac{1}{2}} \approx 0.393$

(b) $P(X=1) = \frac{1}{2} e^{-\frac{1}{2}} \approx 0.303$

(c) $P(X \geq 2) = 1 - P(X=0) - P(X=1) = 1 - e^{-\frac{1}{2}} - \frac{1}{2} e^{-\frac{1}{2}} \approx 0.090$

4.61 Probability of Full House = 0.0014

Let X = # of Full Houses dealt in 1000 hands

then X is binomial with parameters $n = 1000$, $p = 0.0014$

we approximate X by a Poisson R.V. with

$$\lambda = np = 1000 \cdot (0.0014) = 1.4$$

$$P(X \geq 2) = 1 - P(X=0) - P(X=1) = 1 - e^{-1.4} - 1.4e^{-1.4} \approx 0.408$$

Theoretical exercises

4.16 X is Poisson RV with parameter λ

Show $P(X=i)$ increases monotonically and then decreases monotonically, and reaches maximum when $i = \lfloor \lambda \rfloor$

Consider $$\frac{P(X=i)}{P(X=i-1)} = \frac{\lambda^i e^{-\lambda}}{i!} \Big/ \frac{\lambda^{i-1} e^{-\lambda}}{(i-1)!} = \frac{\lambda}{i}$$

So if $i < \lambda$, then $1 < \frac{\lambda}{i}$, and so $P(X=i) > P(X=i-1)$

if $i > \lambda$, then $1 > \frac{\lambda}{i}$, and so $P(X=i) < P(X=i-1)$

So while $i < \lambda$, $P(X=i)$ increases monotonically

and when $i > \lambda$, $P(X=i)$ decreases monotonically

The maximum occurs for that i^* such that $P(X=i^*-1) \leq P(X=i^*) > P(X=i^*+1)$

From the first inequality, we see that $i^* \leq \lambda$
from the second we see $i^*+1 > \lambda$
So so $i^* \leq \lambda < i^*+1$, and hence $i^* = \lfloor \lambda \rfloor$

4.19 Show that if X is a Poisson R.V. w/ parameter λ, then

$$E[X^n] = \lambda E[(X+1)^{n-1}]$$ and compute $E[X^3]$

Proof

$$E[X^n] = \sum_{i=0}^{\infty} i^n \frac{\lambda^i}{i!} e^{-\lambda} = \sum_{i=1}^{\infty} i^n \frac{\lambda^i}{i!} e^{-\lambda}$$

$$= \sum_{i=1}^{\infty} (i)^{n-1} \frac{\lambda^i}{(i-1)!} e^{-\lambda} = \lambda \sum_{i=1}^{\infty} (i)^{n-1} \frac{\lambda^{i-1}}{(i-1)!} e^{-\lambda}$$

$$= \lambda \sum_{j=0}^{\infty} (j+1)^{n-1} \frac{\lambda^j}{j!} e^{-\lambda}$$ Reindexing $j = i-1$

$$= \lambda \sum_{j=0}^{\infty} (j+1)^{n-1} P(X=j) = \lambda E[(X+1)^{n-1}]$$ QED.

So: $E[X] = \lambda E[(X+1)^0] = \lambda E[1] = \lambda \cdot 1 = \lambda$

$$E[X^2] = \lambda E[(X+1)^1] = \lambda(E[X]+1) = \lambda(\lambda+1) = \lambda^2 + \lambda$$

$$E[X^3] = \lambda E[(X+1)^2] = \lambda E[X^2 + 2X + 1]$$

$$= \lambda(E[X^2] + 2E[X] + 1) = \lambda[\lambda(\lambda+1) + 2\lambda + 1] = \lambda^3 + 3\lambda^2 + \lambda$$