Solutions 6
Ch 4 problems

4.13 $P(\text{first sale}) = .3$ Each sale is equally likely to be \$1000 or \$500.

$P(\text{second sale}) = .6$

$X = 0, 500, 1000, 1500, 2000$

$X = 0 \Leftrightarrow$ neither sale is made $P = \underbrace{(1-.3)(1-.6)}_{\text{since sales independent}}$

$p(0) = P(X=0) = (.7)(.4) = .28$

$X = 500 \Leftrightarrow$ exactly one sale of cheap version

$$\underbrace{(.3)}_{\text{first sale}}\underbrace{\left(\tfrac{1}{2}\right)}_{\text{cheap version}}\underbrace{(1-.6)}_{\text{not second sale}} + (1-.3)\underbrace{(.6)}_{\text{second sale}}\tfrac{1}{2}$$

$p(500) = (.3)(.5)(.4) + (.7)(.6)(.5) = .06 + .21 = .27$

$X = 1000 \Leftrightarrow$ one sale of expensive version, or both sales of cheap version

$$p(1000) = \underbrace{(.3)\left(\tfrac{1}{2}\right)(1-.6)}_{\text{first sale expensive}} + \underbrace{(1-.3)(.6)\left(\tfrac{1}{2}\right)}_{\text{second sale expensive}} + \underbrace{(.3)\left(\tfrac{1}{2}\right)(.6)\left(\tfrac{1}{2}\right)}_{\text{both sales cheap}}$$

$= .06 + .21 + .045 = .315$

$X = 1500$: both sales, one expensive, one cheap

$$p(1500) = \underbrace{(.3)\left(\tfrac{1}{2}\right)(.6)\left(\tfrac{1}{2}\right)}_{\text{first sale expensive}} + \underbrace{(.3)\left(\tfrac{1}{2}\right)(.6)\left(\tfrac{1}{2}\right)}_{\text{second sale expensive}}$$

$$= 2(.045) = .09$$

$X = 2000$ both sales expensive:

$$p(2000) = (.3)\left(\tfrac{1}{2}\right)(.6)\left(\tfrac{1}{2}\right) = .045$$

CHECK:

$$.28 + .27 + .315 + .09 + .045 = 1 \checkmark$$

4.18 Four indep coin flips $X = \#$ of heads

$P(X=0) = \left(\tfrac{1}{2}\right)^4 = \tfrac{1}{16}$ $\quad$ $P(X=1) = 4\left(\tfrac{1}{2}\right)^4 = \tfrac{1}{4}$

$P(X=2) = \binom{4}{2}\left(\tfrac{1}{2}\right)^4 = 6\,\tfrac{1}{16} = \tfrac{3}{8}$ $\quad$ $P(X=3) = \binom{4}{3}\left(\tfrac{1}{2}\right)^4 = 4\,\tfrac{1}{16} = \tfrac{1}{4}$

$P(X=4) = \left(\tfrac{1}{2}\right)^4 = \tfrac{1}{16}$

Let $p(a)$ denote the probability mass function of $X-2$

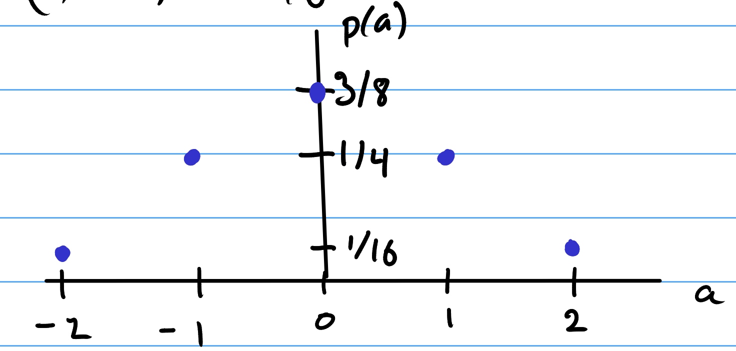
$$p(a) = P(X-2=a) = P(X = a+2)$$

$p(-2) = P(X=0) = 1/16$ $\quad$ $p(-1) = P(X=1) = 1/4$

$p(0) = P(X=2) = 3/8$ $\quad$ $p(1) = P(X=3) = 1/4$

$p(2) = P(X=4) = 1/16$

plot:



4.19 The distribution function of X is given by

$$F(b) = \begin{cases} 0 & b < 0 \\ 1/2 & 0 \le b < 1 \\ 3/5 & 1 \le b < 2 \\ 4/5 & 2 \le b < 3 \\ 9/10 & 3 \le b < 3.5 \\ 1 & 3.5 \le b \end{cases}$$

The probability mass function of X measures the jumps in $F(b)$.

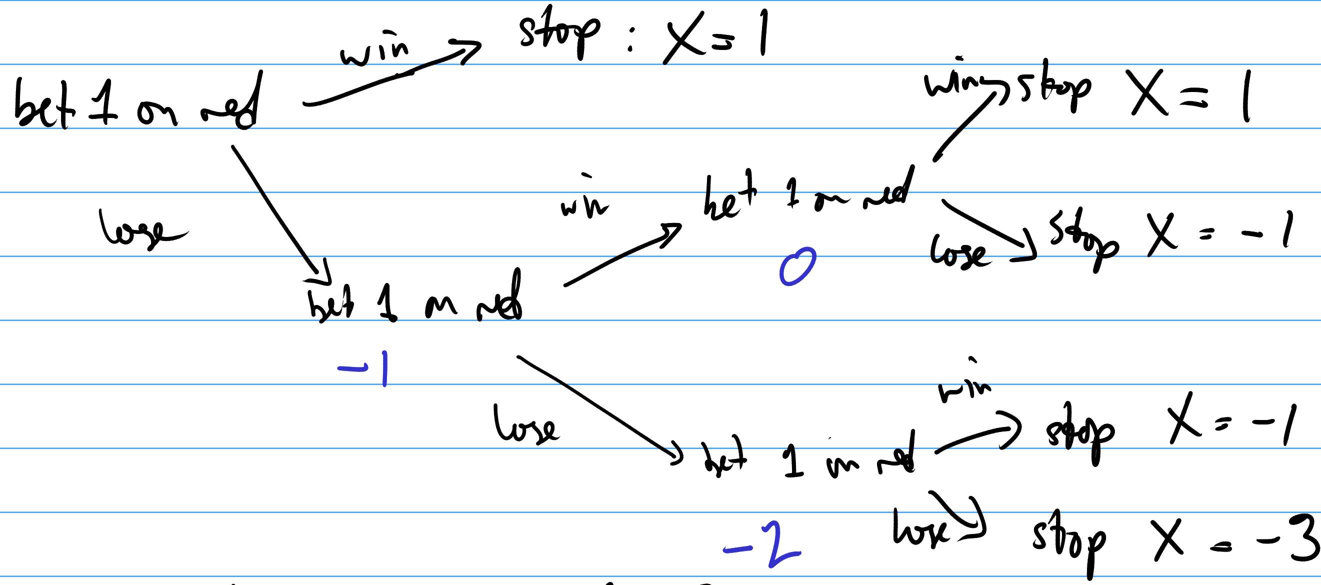
$p(0) = 1/2$, $p(1) = 3/5 - 1/2 = 1/10$,

$p(2) = 4/5 - 3/5 = 1/5$, $p(3) = 9/10 - 4/5 = 1/10$,

$p(3.5) = 1 - 9/10 = 1/10$,

$$p(b) = \begin{cases} 1/2 & b = 0 \\ 1/10 & b = 1 \\ 1/5 & b = 2 \\ 1/10 & b = 3 \\ 1/10 & b = 3.5 \\ 0 & \text{otherwise} \end{cases}$$

Roulette strategy: X = winnings



(a) Possible values: $X = 1, -1, -3$

$$P(X>0) = P(X=1) = P(\text{win}) + P(\text{lose, win, win})$$

$$= \frac{18}{38} + \frac{20}{38}\cdot\frac{18}{38}\cdot\frac{18}{38} = .474 + (.526)\underbrace{(.474)^2}_{.225}$$

$$\approx .592$$

(b) The probability of winning something is greater than $1/2$ so we are more likely to win than lose. But when we win, we get at most 1 dollar, while when we lose, we can lose up to 3 dollars. The expected value will tell the whole story.

(c) $$P(X=-1) = \left(\frac{20}{38}\right)\left(\frac{18}{38}\right)\left(\frac{20}{38}\right) + \left(\frac{20}{38}\right)\left(\frac{20}{38}\right)\left(\frac{18}{38}\right)$$

$$= \frac{14400}{54872} \approx .262$$

$$P(X=-3) = \left(\frac{20}{38}\right)\left(\frac{20}{38}\right)\left(\frac{20}{38}\right) = \frac{8000}{54872} \approx .146$$

$$E[X] \approx 1(.592) + (-1)(.262) + (-3)(.146)$$

$$= .592 - .262 - .438 = -0.108$$

The expected value is negative, so we will lose money in the long run.

4.28 3 items chosen from 20, 4 are defective

X = # of defective items in sample

$$P(X=0) = \binom{16}{3} / \binom{20}{3}$$
$$P(X=1) = \binom{4}{1}\binom{16}{2} / \binom{20}{3}$$
$$P(X=2) = \binom{4}{2}\binom{16}{1} / \binom{20}{3}$$
$$P(X=3) = \binom{4}{3} / \binom{20}{3}$$

$$E[X] = \left[0\binom{16}{3} + 1\binom{4}{1}\binom{16}{2} + 2\binom{4}{2}\binom{16}{1} + 3\binom{4}{3}\right] / \binom{20}{3}$$

$$= [4 \cdot 120 + 2 \cdot 6 \cdot 16 + 3 \cdot 4] / \left[\frac{20 \cdot 19 \cdot 18}{3 \cdot 2}\right]$$

$$= [480 + 192 + 12] / 1140 = 684/1140 = .6$$

Ch 4 Theoretical exercises:

4.2 Suppose X has cumulative distribution function F

So $F(x) = P(X \le x)$

What is distribution function of e^X ?

Let $G(y) = P(e^X \leq y)$ be the distribution function of e^X

$e^X \leq y \iff X \leq \log y$

In this equivalence, we use the fact that log is a monotonically increasing function

Thus $G(y) = P(e^X \leq y) = P(X \leq \log y) = F(\log y)$

So the distribution function of e^X is $F(\log y)$

4.3 If X has distribution function F, what is distribution function of $\alpha X + \beta$, where $\alpha \neq 0$?

Two cases are distinguished $\alpha > 0$ and $\alpha < 0$

Case $\alpha > 0$: $\alpha X + \beta \leq y \iff X \leq \frac{1}{\alpha}(y - \beta)$

$G(y) = P(\alpha X + \beta \leq y) = P(X \leq \frac{1}{\alpha}(y-\beta)) = F\left(\frac{1}{\alpha}(y-\beta)\right)$

is the distribution function of $\alpha X + \beta$

Case $\alpha < 0$: $\alpha X + \beta \leq y \iff X \geq \frac{1}{\alpha}(y - \beta)$

$P(\alpha X + \beta \leq y) = P(X \geq \frac{1}{\alpha}(y-\beta))$ DIVISION BY α REVERSES INEQUALITY

$= 1 - P(X < \frac{1}{\alpha}(y-\beta)) = 1 - \left[P(X \leq \frac{1}{\alpha}(y-\beta)) - P(X = \frac{1}{\alpha}(y-\beta))\right]$

$$= 1 - P\left(X \leq \frac{1}{\alpha}(y-\beta)\right) + P\left(X = \frac{1}{\alpha}(y-\beta)\right)$$

$$= 1 - F\left(\frac{1}{\alpha}(y-\beta)\right) + p\left(\frac{1}{\alpha}(y-\beta)\right)$$

where F is the cumulative distribution function of X and p is the probability mass function of X

$$G(y) = 1 - F\left(\frac{1}{\alpha}(y-\beta)\right) + p\left(\frac{1}{\alpha}(y-\beta)\right)$$

If α is negative.

4.4 Let N be a random variable whose values are nonnegative integers.

Want to show $E[N] = \sum_{i=1}^{\infty} P\{N \geq i\}$

Using $P\{N \geq i\} = \sum_{k=i}^{\infty} P\{N = k\}$

gives $\sum_{i=1}^{\infty} P\{N \geq i\} = \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} P\{N = k\}$

this is a double sum over pairs (i, k) such that $k \geq i$

$$\begin{array}{c|cccc} & \xrightarrow{k} & & & \\ i \downarrow & (1,1) & (1,2) & (1,3) & \cdots \\ & & (2,2) & (2,3) & \cdots \\ & & & (3,3) & \cdots \\ & & & & \ddots \end{array}$$

where k is summed over first

Summing over i first gives $\sum_{k=1}^{\infty} \sum_{i=1}^{k}$ (stuff)

Thus $\sum_{i=1}^{\infty} \sum_{k=i}^{\infty} P\{N=k\} = \sum_{k=1}^{\infty} \sum_{i=1}^{k} P\{N=k\}$

$= \sum_{k=1}^{\infty} k P\{N=k\} = E[N]$ by definition of expectation.

Technical point: reversing the order of summation is legitimate because the sum consists of positive terms: The existance of $E[N]$ (which is tacitly assumed) implies convergence of the sum in the order $\sum_k \sum_i$ (stuff)

since the terms are positive, the convergence is absolute, and so the rearranged sum $\sum_i \sum_k$ (stuff) also converges

and has the same value.

4.7 X a random variable with $E[X] = \mu$ and $Var(X) = \sigma^2$

Find expectation and variance of $Y = \frac{X-\mu}{\sigma}$

$E[Y] = E\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}\left(E[X] - \mu\right)$ by Corollary 4.1

$= \frac{1}{\sigma}(\mu - \mu) = 0$

$Var(Y) = E\left[\left(Y - E[Y]\right)^2\right] = E[Y^2]$

$E\left[\frac{(X-\mu)^2}{\sigma^2}\right] = \frac{1}{\sigma^2} E\left[(X-\mu)^2\right] = \frac{1}{\sigma^2} Var(X) = \frac{\sigma^2}{\sigma^2} = 1$

4.8 Find $Var(X)$ if $P(X=a) = p = 1 - P(X=b)$

$$E[X] = a\,P(X=a) + b\,P(X=b) = ap + b(1-p)$$

$$= b + (a-b)p$$

$$Var(X) = E[X^2] - E[X]^2$$

$$E[X^2] = a^2 p + b^2(1-p)$$

$$Var(X) = a^2 p + b^2(1-p) - (b + (a-b)p)^2$$