Solutions 5

Ch 3 p. 113

3.26 (5.11) is $P(E_1 | E_2 F) = P(E_1 | F)$

(5.12) is $P(E_1 E_2 | F) = P(E_1 | F) P(E_2 | F)$

These equations are equivalent

Suppose (5.11) holds: multiply both sides by $P(E_2 | F)$

$$P(E_1 | F) P(E_2 | F) \overset{(5.11)}{=} P(E_1 | E_2 F) P(E_2 | F)$$

$$= \frac{P(E_1 E_2 F)}{P(E_2 F)} \frac{P(E_2 F)}{P(F)}$$ by definition of conditional Prob.

$$= \frac{P(E_1 E_2 F)}{P(F)} = P(E_1 E_2 | F)$$

So (5.12) holds as well

Supposing (5.12) holds, divide both sides by $P(E_2 | F)$ and use a similar argument to obtain (5.11)

Alternatively: $Q(E) = P(E|F)$ then $Q$ is a probability

$$\left.\begin{array}{ll} (5.11) & Q(E_1 | E_2) = Q(E_1) \\ (5.12) & Q(E_1 E_2) = Q(E_1) Q(E_2) \end{array}\right\}$$ Know these are equivalent notions of independence.

3.28 Prove or give counterexample:

if $E_1$ and $E_2$ are independent, then they are conditionally independent given $F$.

Flip coin 2 times: $E_1$ = first is H
$E_2$ = second is H

$F = E_1 \cup E_2$ = first or second is H.

$P(E_1) = 1/2 \quad P(E_2) = 1/2 \quad P(F) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$

Then $E_1$ and $E_2$ are independent, BUT:

$$P(E_1|F) = \frac{P(E_1(E_1 \cup E_2))}{P(F)} = \frac{P(E_1)}{P(F)} = \frac{1/2}{3/4} = \frac{2}{3}$$

$P(E_2|F) = 2/3$ as well

$$P(E_1E_2|F) = \frac{P(E_1E_2(E_1 \cup E_2))}{P(F)} = \frac{P(E_1E_2)}{P(F)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Cond. Indep. means: $P(E_1E_2|F) = P(E_1|F)P(E_2|F)$

$\frac{1}{3} \neq \left(\frac{2}{3}\right)\cdot\left(\frac{2}{3}\right)$ so $E_1$ and $E_2$ are not conditionally independent, given $F$.

Recall:

3.29 $C_i$ = ith coin selected

$F_n$ = first n flips are Heads

Let $H_m$ = next m flips are Heads

$$P(H_m|F_n) = \sum_{i=0}^{k} P(H_m|F_nC_i)\,P(C_i|F_n)$$

Just as before, we know $P(C_i|F_n) = \dfrac{(i/k)^n}{\sum_{j=0}^{k}(j/k)^n}$

Now $P(H_m|F_nC_i) = P(H_m|C_i) = (i/k)^m$,

since flips are conditionally independent, given $C_i$

$$\text{So } P(H_m|F_n) = \sum_{i=0}^{k} (i/k)^m \frac{(i/k)^n}{\sum_{j=0}^{k}(j/k)^n}$$

$$= \frac{\sum_{i=0}^{k}(i/k)^{n+m}}{\sum_{j=0}^{k}(j/k)^n} = \frac{\frac{1}{k}\sum_{i=0}^{k}(i/k)^{n+m}}{\frac{1}{k}\sum_{j=0}^{k}(j/k)^n}$$

As $k \to \infty$, $\frac{1}{k}\sum_{i=0}^{k}(i/k)^{n+m} \to \int_0^1 x^{n+m}\,dx = \frac{1}{n+m+1}$

$\frac{1}{k}\sum_{j=0}^{k}(j/k)^n \to \int_0^1 x^n\,dx = \frac{1}{n+1}$

So $P(H_m|F_n) \to \dfrac{n+1}{n+m+1}$

Ch 4 Problems pp. 172 – 173

4.1 Two balls chosen from

| | win |
|---|---|
| 8 white | $-\$1$ |
| 4 black | $+\$2$ |
| 2 orange | 0 |

$X$ = winnings. Possible Values: $-2, -1, 0, 1, 2, 4$

It is impossible to win exactly 3 if we draw two balls

There are $\binom{14}{2}$ possible outcomes

$P(X=-2) = \dfrac{\binom{8}{2}}{\binom{14}{2}}$ both white

$P(X=-1) = \dfrac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}}$ one white, one orange

$P(X=0) = \dfrac{\binom{2}{2}}{\binom{14}{2}}$ both orange

$P(X=1) = \dfrac{\binom{8}{1}\binom{4}{1}}{\binom{14}{2}}$ one white and one black

$P(X=2) = \dfrac{\binom{4}{1}\binom{2}{1}}{\binom{14}{2}}$ one black and one orange

$P(X=4) = \dfrac{\binom{4}{2}}{\binom{14}{2}}$ both black

4.3 Three dice rolled $6^3 = 216$ equally likely outcomes

$X$ = sum of three dice

$X$ can take any of the values 3 – 18

| Value | Outcomes | Prob |
|---|---|---|
| 3 | (1,1,1) | 1/216 |
| 4 | (1,1,2) in 3 orders | 3/216 |
| 5 | (1,1,3) in 3 ways<br>(1,2,2) in 3 ways | 6/216 |
| 6 | (1,1,4) in 3 ways<br>(1,2,3) in 6 ways<br>(2,2,2) in 1 way | 10/216 |
| 7 | (1,1,5) in 3<br>(1,2,4) in 6<br>(1,3,3) in 3<br>(2,2,3) in 3 | 15/216 |
| 8 | (1,1,6) in 3<br>(1,2,5) in 6<br>(1,3,4) in 6<br>(2,2,4) in 3<br>(2,3,3) in 3 | 21/216 |

| | | | |
|---|---|---|---|
| 9 | (1, 2, 6) | in 6 | |
| | (1, 3, 5) | in 6 | |
| | (1, 4, 4) | in 3 | 25/216 |
| | (2, 2, 5) | in 3 | |
| | (2, 3, 4) | in 6 | |
| | (3, 3, 3) | in 1 | |
| 10 | (1, 3, 6) | in 6 | |
| | (1, 4, 5) | in 6 | 27/216 |
| | (2, 2, 6) | in 3 | |
| | (2, 3, 5) | in 6 | |
| | (2, 4, 4) | in 3 | |
| | (3, 3, 4) | in 3 | |
| 11 | (1, 4, 6) | in 6 | |
| | (1, 5, 5) | in 3 | |
| | (2, 3, 6) | in 6 | 27/216 |
| | (2, 4, 5) | in 6 | |
| | (3, 3, 5) | in 3 | |
| | (3, 4, 4) | in 3 | |
| 12 | (1, 5, 6) | in 6 | |
| | (2, 4, 6) | in 6 | |
| | (2, 5, 5) | in 3 | 25/216 |
| | (3, 3, 6) | in 3 | |
| | (3, 4, 5) | in 6 | |
| | (4, 4, 4) | in 1 | |

| | | | |
|---|---|---|---|
| 13 | (1, 6, 6) | is 3 | |
| | (2, 5, 6) | is 6 | |
| | (3, 4, 6) | is 6 | 21/216 |
| | (3, 5, 5) | is 3 | |
| | (4, 4, 5) | is 3 | |
| 14 | (2, 6, 6) | is 3 | |
| | (3, 5, 6) | is 6 | 15/216 |
| | (4, 4, 6) | is 3 | |
| | (4, 5, 5) | is 3 | |
| 15 | (3, 6, 6) | is 3 | 10/216 |
| | (4, 5, 6) | is 6 | |
| | (5, 5, 5) | is 1 | |
| 16 | (4, 6, 6) | is 3 | 6/216 |
| | (5, 5, 6) | is 3 | |
| 17 | (5, 6, 6) | is 3 | 3/216 |
| 18 | (6, 6, 6) | is 1 | 1/216 |

Note symmetry : $P(X=n) = P(X=21-n)$

Check: $(1+3+6+10+15+21+25+27)\times 2$

$= 108\times 2 = 216$ So Probs add up to 1.

4.5 Coin is tossed $n$ times

$X =$ # of heads $-$ # tails

Possible values: All heads : $X = n$

one tail : $X = (n-1) - 1 = n-2$

two tails : $X = (n-2) - 2 = n-4$

In general $X = n - 2(\#\text{ tails})$ so

$X = n, n-2, n-4, \ldots, -n+4, -n+2, -n$

If we list possible values in decreasing order.

4.6 Assume $n=3$ and coin is fair.

So $X = -3, -1, 1, 3$ are possible

$P(X=-3) = P(\{TTT\}) = 1/8$

$P(X=-1) = P(\{HTT, THT, TTH\}) = 3/8$

$P(X=1) = P(\{HHT, HTH, THH\}) = 3/8$

$P(X=3) = P(\{HHH\}) = 1/8$