Solutions 4

Ch 3 Problems

3.12 Let E_1, E_2, E_3 denote events that student passes 1st, 2nd, and 3rd exams

So $P(E_1) = .9 \quad P(E_2|E_1) = .8 \quad P(E_3|E_1E_2) = .7$

Passes 1st & 2nd: $P(E_2E_1) = P(E_2|E_1)P(E_1) = (.8)(.9) = .72$

(a) Pass All $P(E_3E_2E_1) = P(E_3|E_1E_2)P(E_1E_2) = (.7)(.72) = .504$

(b) P(failed second | did not pass all)

$$= P\left(E_1E_2^c \,|\, (E_1E_2E_3)^c\right) = \frac{P\left((E_1E_2E_3)^c \,|\, E_1E_2^c\right)P(E_1E_2^c)}{P((E_1E_2E_3)^c)}$$

$$P\left((E_1E_2E_3)^c \,|\, E_1E_2^c\right) = 1$$

(if she fails second exam, she cannot pass all 3)

$$P((E_1E_2E_3)^c) = 1 - P(E_1E_2E_3) = 1 - .504 = .496$$

$$P(E_1E_2^c) = P(E_1) - P(E_1E_2)$$ since $E_1 = E_1E_2 \cup E_1E_2^c$ mutually exclusive

$$= .9 - .72 = .18$$

so Answer $= \dfrac{(1)(.18)}{.496} = .362904$

3.22 Roll red, blue, yellow dice

If R, B, Y are numbers appearing on the dice, we ask for the probability of the event that $B < Y < R$.

(a) P(no two dice on same #)

Sample space has 6^3 equally likely outcomes

There are $\underset{B}{6} \cdot \underset{Y}{5} \cdot \underset{R}{4}$ outcomes in which no number is repeated

so $P(B \neq Y \text{ and } Y \neq R \text{ and } B \neq R) = \dfrac{6 \cdot 5 \cdot 4}{6^3} = \dfrac{10}{18}$

(b) Given no two dice land on same #, what is probability that $B < Y < R$

The reduced sample space consists of $6 \cdot 5 \cdot 4$ outcomes, all equally likely. There are $\binom{6}{3} = \dfrac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1}$ possible sets of numbers that can appear, and each set of numbers can be distributed across the Blue, Yellow, Red dice in $3 \cdot 2 \cdot 1 = 6$ ways. For a given set of 3 numbers there is only one of these 6 ways that satisfies $B < Y < R$, so

$$P(B<Y<R \mid \text{no two same}) = \frac{\left(\frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1}\right)}{6 \cdot 5 \cdot 4} = \frac{1}{3 \cdot 2 \cdot 1} = \frac{1}{6}.$$

(c) $P(B<Y<R) = P(B<Y<R \mid \text{no two same})\, P(\text{no two same}) = \dfrac{10}{18} \cdot \dfrac{1}{6} = \dfrac{10}{108}$

3.24 2 balls are either gold or black w/ $\frac{1}{2}$ probability, and independently from each other, placed in urn

4 outcomes each with $\frac{1}{4}$ probability: $\begin{cases} GG \\ GB \\ BG \\ BB \end{cases}$

(a) We know at least one ball is Gold $F = \{GG, GB, BG\}$
Both gold $= \{GG\}$
$P(\text{Both gold} \mid \text{at least one gold}) = P(\{GG\} \mid \{GG, GB, BG\})$

$$= \frac{P(\{GG\})}{P(\{GG, GB, BG\})} = \frac{1/4}{3/4} = \frac{1}{3}$$

(b) A gold ball falls out of urn, so the probability of the other being gold is $\frac{1}{2}$.

$P(\text{both gold} \mid \text{discovered ball is gold}) = \frac{1}{2}$

This is an apparent paradox, since it seems in both cases that we know one of the balls is gold. But in the second case, we also have to take into account the chance that the other ball could have fallen out.
So in (b) there are actually 8 possible outcomes

GG	GG	O = which one falls out.
GB	GB	
BG	BG	
BB	BB	

So we see that exactly half the cases where a gold ball falls out, the other ball is gold.

3.37 Gambler has fair coin $(P(H)=\frac{1}{2}=P(T))$
and a 2-head coin $(P(H)=1,\ P(T)=0)$

(a) one coin is selected at random, flipped, comes up heads what is probability the coin is fair.

$$P(\text{fair}|H) = \frac{P(H|\text{fair})\,P(\text{fair})}{P(H|\text{fair})\,P(\text{fair}) + P(H|\text{2-headed})\,P(\text{2-headed})}$$

$P(\text{fair}) = P(\text{2-headed}) = \frac{1}{2}$ since coin selected randomly

$$P(\text{fair}|H) = \frac{(\frac{1}{2})(\frac{1}{2})}{(\frac{1}{2})(\frac{1}{2}) + (1)(\frac{1}{2})} = \frac{1/4}{1/4+1/2} = \frac{1/4}{3/4} = \frac{1}{3}$$

(b) coin flipped again, heads again, what prob. coin is fair.

$$P(\text{fair}|HH) = \frac{P(HH|\text{fair})\,P(\text{fair})}{P(HH|\text{fair})\,P(\text{fair}) + P(HH|\text{2-headed})\,P(\text{2-headed})}$$

$$= \frac{(\frac{1}{4})(\frac{1}{2})}{(\frac{1}{4})(\frac{1}{2}) + (1)(\frac{1}{2})} = \frac{1/8}{1/8+1/2} = \frac{1/8}{5/8} = \frac{1}{5}$$

(c) Coin fliped, comes up tails what is probability coin is fair
Obviously the coin must be fair:

$$P(\text{fair}|HHT) = \frac{P(HHT|\text{fair})\,P(\text{fair})}{P(HHT|\text{fair})\,P(\text{fair}) + P(HHT|\text{2-head})\,P(\text{2-head})}$$

Since $P(HHT \mid 2\text{-headed}) = 0$, get

$P(\text{fair} \mid HHT) = 1$, as expected.

3.47 Urn w/ 5 white, 10 black.

Fair die is rolled, that number of balls is drawn

P(all white?)

Let R_i $(i=1,2,3,4,5,6)$ denote the event that the die rolls the number i

W = event that all balls drawn are white

$$P(W) = \sum_{i=1}^{6} P(W \mid R_i) P(R_i), \quad P(R_i) = \frac{1}{6}$$

$$P(W \mid R_i) = \text{prob. of getting } i \text{ white when drawing } i = \frac{\binom{5}{i}}{\binom{15}{i}}$$

$$P(W) = \frac{\binom{5}{1}}{\binom{15}{1}}\frac{1}{6} + \frac{\binom{5}{2}}{\binom{15}{2}}\frac{1}{6} + \frac{\binom{5}{3}}{\binom{15}{3}}\frac{1}{6} + \frac{\binom{5}{4}}{\binom{15}{4}}\frac{1}{6} + \frac{\binom{5}{5}}{\binom{15}{5}}\frac{1}{6} + \frac{\binom{5}{6}}{\binom{15}{6}}\frac{1}{6} \qquad \binom{5}{6} = 0$$

$$= \frac{1}{6}\left[\frac{5}{15} + \frac{5\cdot4}{15\cdot14} + \frac{5\cdot4\cdot3}{15\cdot14\cdot13} + \frac{5\cdot4\cdot3\cdot2}{15\cdot14\cdot13\cdot12} + \frac{5\cdot4\cdot3\cdot2\cdot1}{15\cdot14\cdot13\cdot12\cdot11} + 0\right]$$

$$= \frac{1}{6}\cdot\frac{5}{11}$$

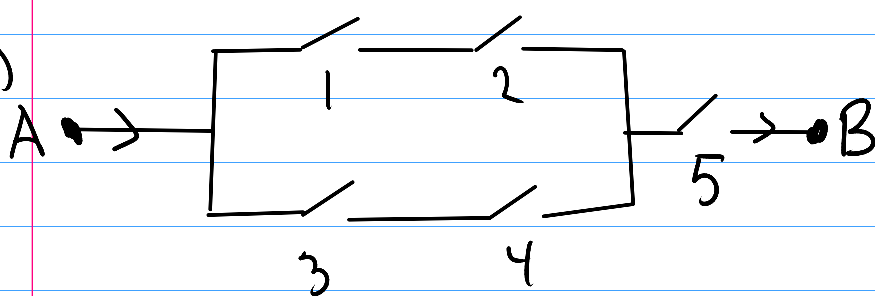
$$P(R_3 \mid W) = \frac{P(W \mid R_3)\,P(R_3)}{P(W)}$$

$$= \left[\frac{\binom{5}{3}}{\binom{15}{3}}\right] \cdot \frac{1}{6} \Big/ P(W) = \left(\frac{2}{91}\right) \cdot \frac{1}{6} \Big/ \left(\frac{5}{11}\right) \cdot \frac{1}{6}$$

$$= \frac{22}{455}$$

3.66 Let E_i denote the event that ith relay closes.
These events are independent and $P(E_i) = p_i$

(a)



Current flows $\Leftrightarrow$ (1 & 2 close or 3 & 4 close) and 5 closes

$$= (E_1E_2 \cup E_3E_4)E_5$$

$$P((E_1E_2 \cup E_3E_4)E_5) = P(E_1E_2 \cup E_3E_4)\,P(E_5) \text{ by independence}$$

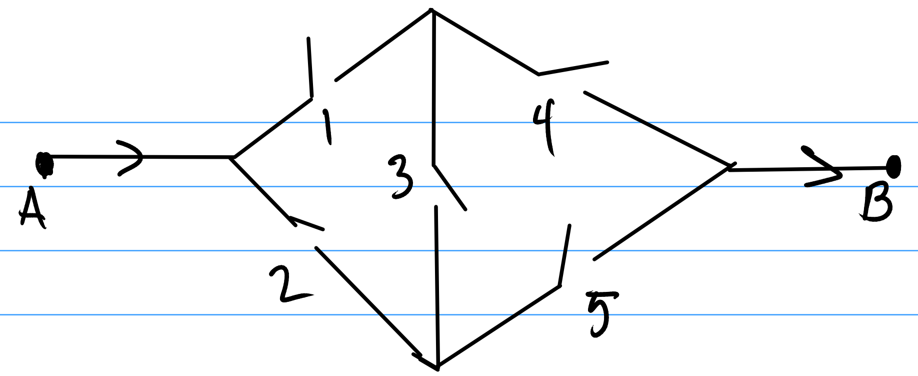
$$P(E_1E_2 \cup E_3E_4) = P(E_1E_2) + P(E_3E_4) - P(E_1E_2E_3E_4)$$

by inclusion exclusion

$$= p_1p_2 + p_3p_4 - p_1p_2p_3p_4 \quad \text{by independence}$$

$$P(\text{current flows}) = [p_1p_2 + p_3p_4 - p_1p_2p_3p_4]\,p_5$$

(b)



Condition on E_3:

If 3 closes, need (1 or 2) and (4 or 5)

$$P(\text{current flows} \mid E_3) = P\big((E_1 \cup E_2)(E_4 \cup E_5)\big)$$

$$= P(E_1 \cup E_2)\, P(E_4 \cup E_5) \quad \text{by independence}$$

$$= \big[P(E_1) + P(E_2) - P(E_1E_2)\big]\big[P(E_4) + P(E_5) - P(E_4E_5)\big]$$

$$= \big[p_1 + p_2 - p_1p_2\big]\big[p_4 + p_5 - p_4p_5\big]$$

If 3 is open, need (1 and 4) or (2 and 5)

$$P(\text{current flows} \mid E_3^c) = P(E_1E_4 \cup E_2E_5)$$

$$= P(E_1E_4) + P(E_2E_5) - P(E_1E_4E_2E_5)$$

$$= p_1p_4 + p_2p_5 - p_1p_2p_4p_5$$

$$P(\text{current flows}) = P(\text{current flows} \mid E_3)P(E_3) + P(\text{current flows} \mid E_3^c)P(E_3^c)$$

$$= \big[p_1 + p_2 - p_1p_2\big]\big[p_4 + p_5 - p_4p_5\big]p_3 + \big[p_1p_4 + p_2p_5 - p_1p_2p_4p_5\big](1 - p_3)$$

3.78 A and B play series of games
Each game is independent, $P(A \text{ wins}) = p$
$P(B \text{ wins}) = 1 - p$

Match ends when one player has won 2 more games than the other has won.

Note that a match always has an even # of games: A wins n times, B wins n+2 times, or vice versa, so 2n+2 games are played

(a) $P(\text{4 games are played})$

length of match:	2	4	Here, "A" denotes a win for A, B same.
possible sequence of wins	AA	ABAA	
	BB	ABBB	
		BAAA	
		BABB	

$$P(\text{Match lasts 4 games}) = P(ABAA) + P(ABBB) + P(BAAA) + P(BABB)$$
$$= p^3(1-p) + p(1-p)^3 + p^3(1-p) + p(1-p)^3$$
$$= 2\left[p^3(1-p) + p(1-p)^3\right]$$

(b) $P(\text{A wins match})$

The rules of this game are similar to the rules for a tie game of Tennis ("deuce"/"advantage" system)

We group the games into consecutive pairs:
Starting from a tie 4 things can happen

AA $\quad$ Tie $\xrightarrow{A}$ advantage A $\xrightarrow{A}$ A wins match

AB $\quad$ Tie $\xrightarrow{A}$ advantage A $\xrightarrow{B}$ tie

BA $\quad$ Tie $\xrightarrow{B}$ advantage B $\xrightarrow{A}$ tie

BB $\quad$ Tie $\xrightarrow{B}$ advantage B $\xrightarrow{B}$ B wins match.

So starting from tie, each sequence of two games results in either a tie or a win for one side.

So A wins $\iff$ some sequence of BA or AB pairs, followed by AA

Let E_i = AB or BA on games $2i-1$ and $2i$

F_i = AA on games $2i-1$ and $2i$

$$P(E_i) = \underbrace{p(1-p)}_{A\ B} + \underbrace{(1-p)p}_{B\ A} = 2p(1-p)$$

$$P(F_i) = p^2$$

These events are independent for different values of i.

A wins in $2n+2$ games $= E_1 E_2 \cdots E_n F_{n+1}$

$$P(\text{A wins in } 2n+2 \text{ games}) = [2p(1-p)]^n p^2$$

$$\{A \text{ wins}\} = \bigcup_{n=0}^{\infty} \{A \text{ wins in } 2n+2 \text{ games}\}$$

$$P(A \text{ wins}) = \sum_{n=0}^{\infty} [2p(1-p)]^n p^2$$

$$= p^2 \sum_{n=0}^{\infty} [2p(1-p)]^n = p^2 \frac{1}{1-(2p(1-p))} = \frac{p^2}{1-2p+2p^2}$$

$$= \frac{p^2}{p^2+(1-p)^2}.$$

3.4 Ball in one of n boxes $P(\text{in } i\text{th box}) = P_i$
search of ith box finds it with probability α_i

Let E_i = ball is in ith box
S_i = search of ith box finds ball.

$$P(E_j \mid S_i^c) = \frac{P(S_i^c \mid E_j)\, P(E_j)}{\sum_{k=1}^{n} P(S_i^c \mid E_k) P(E_k)}$$

$$P(E_j) = P_j \qquad P(S_i^c \mid E_j) = \begin{cases} 1 & \text{if } i \neq j \\ 1-\alpha_i & \text{if } i = j \end{cases}$$

Thus $\displaystyle\sum_{k=1}^{n} P(S_i^c \mid E_k) P(E_k) = \sum_{k \neq i} P_k + (1-\alpha_i) P_i$

$$= \left(\sum_{k=1}^{n} P_k\right) - \alpha_i P_i = 1 - \alpha_i P_i$$

So $P(E_j | S_i^c) = \dfrac{P(S_i^c | E_j) P_j}{1 - \alpha_i P_i} = \begin{cases} \dfrac{P_j}{1-\alpha_i P_i} & j \neq i \\ \dfrac{(1-\alpha_i) P_i}{1 - \alpha_i P_i} & j = i \end{cases}$ $\boxtimes$

3.6 Suppose $E_1, E_2, E_3, \dots, E_n$ are independent

$P(E_1 \cup E_2 \cup \cdots \cup E_n) = P([E_1^c E_2^c \cdots E_n^c]^c)$ by DeMorgan

$= 1 - P(E_1^c E_2^c \cdots E_n^c)$

$= 1 - \prod_{i=1}^{n} P(E_i^c)$ by independence

$= 1 - \prod_{i=1}^{n} [1 - P(E_i)]$

3.10 n people $A_{i,j}$ = persons i and j have same birthday

$P(A_{i,j}) = \dfrac{1}{365}$

The events $A_{i,j}$ & $A_{r,s}$ for different pairs are independent

if $\{i,j\} \cap \{r,s\} = \emptyset$ this is obvious

otherwise, we are looking at $A_{i,j}$ and $A_{j,k}$

$P(A_{j,k} | A_{i,j})$ = Prob that k has birthday on the particular day which i and j share =

$$= \frac{1}{365} = P(A_{j,k})$$

But what about the three events $A_{i,j}, A_{j,k}, A_{i,k}$

If these are to be independent, we need

$$P(A_{i,k} \mid A_{i,j} A_{j,k}) = \frac{1}{365}, \quad \text{But No!}$$

i and j share birthday, and j and k share birthday, then i and k share birthday, necessarily

so in fact $P(A_{i,k} \mid A_{i,j} A_{j,k}) = 1 \neq \frac{1}{365}$

And these events are not independent all together.

3.16 Bernoulli trials: $P_n = P(\text{n trials result in even \# of successes})$

E_n = n trials result in even # of successes
E_n^c = n trials result in odd # of successes

S_i = success on the ith trial

condition on S_1 (whether first trial is success)

$$P_n = P(E_n) = P(E_n \mid S_1)P(S_1) + P(E_n \mid S_1^c)P(S_1^c)$$

$P(E_n \mid S_1) = P(E_{n-1}^c)$ b/c need odd # of successes in next n-1 trials

$P(E_n \mid S_1^c) = P(E_{n-1})$ b/c need even # of successes in n-1 trials

So $P_n = (1-P_{n-1})p + P_{n-1}(1-p)$

Claim: $P_n = \frac{1+(1-2p)^n}{2}$

Base case:

$P_1 = P(S_1^c) = (1-p) = \frac{1+(1-2p)^1}{2}$ ✓

Induction step: Suppose $P_{n-1} = \frac{1+(1-2p)^{n-1}}{2}$

$P_n = p(1-P_{n-1}) + (1-p)P_{n-1} = p + P_{n-1} - 2pP_{n-1}$

$= p + \frac{1+(1-2p)^{n-1}}{2} - 2p\frac{1+(1-2p)^{n-1}}{2}$

$= \frac{1}{2}\left[2p+1+(1-2p)^{n-1} - 2p\left(1+(1-2p)^{n-1}\right)\right]$

$= \frac{1}{2}\left[\cancel{2p}+1+(1-2p)^{n-1} \cancel{-2p} - 2p(1-2p)^{n-1}\right]$

$= \frac{1}{2}\left[1+(1-2p)(1-2p)^{n-1}\right] = \frac{1}{2}\left[1+(1-2p)^n\right]$ ✓

QED.

3.18 E_n = no run of 3 consecutive Heads in n coin flips

$Q_n = P(E_n)$

Clearly $Q_0 = Q_1 = Q_2 = 1$

Our sample space may be broken into 4 parts, depending on how the sequence starts

$F_1 = \{T \cdots\}$ $\qquad P(F_1) = 1/2$

$F_2 = \{HT \cdots\}$ $\qquad P(F_2) = 1/4$

$F_3 = \{HHT \cdots\}$ $\qquad P(F_3) = 1/8$

$F_4 = \{HHH \cdots\}$ $\qquad P(F_4) = 1/8$

$$Q_n = P(E_n|F_1)P(F_1) + P(E_n|F_2)P(F_2) + P(E_n|F_3)P(F_3) + P(E_n|F_4)P(F_4)$$

$P(E_n|F_1) = P($no 3 consecutive H on $n-1$ trials$) = Q_{n-1}$

$P(E_n|F_2) = P($no 3 consecutive H on $n-2$ trials$) = Q_{n-2}$

$P(E_n|F_3) = Q_{n-3}$ (similarly)

$P(E_n|F_4) = 0$ since $F_3 \Rightarrow$ 3 consecutive heads

so $Q_n = \frac{1}{2}Q_{n-1} + \frac{1}{4}Q_{n-2} + \frac{1}{8}Q_{n-3}$

$Q_0 = Q_1 = Q_2 = 1$

$Q_3 = \frac{7}{8}$ $\qquad Q_4 = \frac{1}{2}\frac{7}{8} + \frac{1}{4} + \frac{1}{8} = \frac{13}{16}$

$Q_5 = \frac{1}{2}\frac{13}{16} + \frac{1}{4}\frac{7}{8} + \frac{1}{8} = \frac{24}{32}$

$Q_6 = \frac{1}{2}\frac{24}{32} + \frac{1}{4}\frac{13}{16} + \frac{1}{8}\frac{7}{8} = \frac{44}{64}$

$Q_7 = \frac{1}{2}\frac{44}{64} + \frac{1}{4}\frac{24}{32} + \frac{1}{8}\frac{13}{16} = \frac{81}{128}$

$Q_8 = \frac{1}{2}\frac{81}{128} + \frac{1}{4}\frac{44}{64} + \frac{1}{8}\frac{24}{32} = \frac{149}{256}$