Solutions 3
Ch 2 Problems

8. A, B mutually exclusive $P(A) = .3$ $P(B) = .5$

(a) $P(A \cup B) = P(A) + P(B) - P(AB) = .3 + .5 - 0 = .8$

Since $P(AB) = P(\emptyset) = 0$

(b) A but not B $= AB^c$

Now $A = AS = A(B \cup B^c) = AB \cup AB^c = \emptyset \cup AB^c = AB^c$

$A = AB^c$

$P(AB^c) = P(A) = .3$

(c) both A and B: $P(AB) = P(\emptyset) = 0$

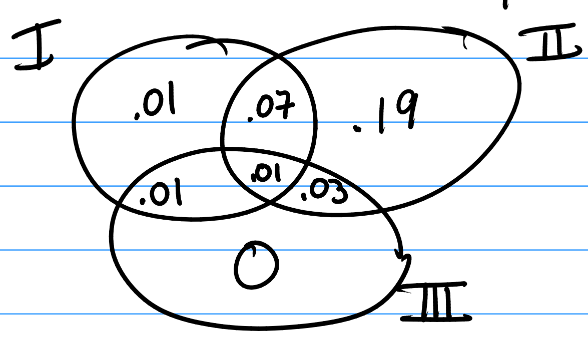
13. Let E_I, E_{II} and E_{III} denote events that a random person reads each of the newspapers I, II, III

So we know $P(E_I) = .1$ $P(E_{II}) = .3$ $P(E_{III}) = .05$

$P(E_I E_{II}) = .08$ $P(E_I E_{III}) = .02$ $P(E_{II} E_{III}) = .04$

$P(E_I E_{II} E_{III}) = .01$

We can solve for all parts of the venn diagram



e.g.

$P(E_I E_{II} E_{III}^c) = P(E_I E_{II}) - P(E_I E_{II} E_{III})$

$= .07$

(a) $P(\text{read } \underline{\text{only}} \text{ one}) = .01 + .19 + 0 = .20$

$.20 \times 100000 = 20000$ people read only one

(b) $P(\text{read at least } 2) = .07 + .01 + .01 + .03 = .12$

$.12 \times 100000 = 12000$ people read at least 2

(c) $P(\text{I or III and II}) = P((E_{I} \cup E_{III})E_{II})$

$= P(E_{I}E_{II} \cup E_{II}E_{III}) = .07 + .03 + .01 = .11$

$.11 \times 100000 = 11000$ people read (I or III) and II

(d) $P((E_{I} \cup E_{II} \cup E_{III})^{c}) = 1 - P(E_{I} \cup E_{II} \cup E_{III})$

$= 1 - [.01 + .07 + .19 + .01 + .01 + .03] = 1 - .32 = .68$

$.68 \times 100000 = 68000$ people do not read.

(e) $P((\text{I or III but not both}) \text{ and II}) = .07 + .03 = .10$

$.10 \times 100000 = 10000$ people

18. Two cards selected from 52-card deck
Blackjack = A and (10, J, Q, or K)

There are 4 aces in the deck and 16 (10, J, Q or K)'s so there are $4 \cdot 16$ Blackjack hands

If we assume all $\binom{52}{2}$ hands are equally likely, we get

$$P(\text{Blackjack}) = \frac{4 \cdot 16}{\binom{52}{2}} \approx 0.048$$

22. Shuffle n cards: flip coin n times: if H leave card and move on
if T, move current card to back of deck and move on.

$S = \{$ all H-T sequences of length $n\}$ # outcomes in $S = 2^n$

Outcomes which leave deck unchanged:

$$\left.\begin{array}{l} HH \cdots HHH \\ HH \cdots HHT \\ HH \cdots HTT \\ \vdots \\ HT \cdots TTT \\ TT \cdots TTT \end{array}\right\}$$

in general: get heads for a while, deck doesn't change up to that point. After first tails, must get all tails from then on so that all cards cycle through back to original position.

Such a sequence consists of k heads in a row, followed by $n-k$ tails in a row $k = 0, \ldots, n$

So there are $n+1$ such sequences

$$P(\text{deck ends up in same order}) = \frac{n+1}{2^n}$$

25. A pair of dice is rolled until a sum of 5 or 7
Find probability that 5 occurs first.

Consider just one trial, where two dice are thrown
36 total outcomes.
4 outcomes result in a sum of 5
6 outcomes result in a sum of 7
26 outcomes result in neither 5 nor 7.

E_n = no 5 or 7 on first n-1 rolls, 5 on nth roll.

$$P(E_n) = \underbrace{\frac{26}{36}\cdot\frac{26}{36}\cdot \cdots \cdot\frac{26}{36}}_{n-1}\cdot\frac{4}{36} = \left(\frac{26}{36}\right)^{n-1}\frac{4}{36}$$

Let E = 5 occurs before 7
If an outcome x is in E, the 5 must occur on the nth roll (for some value of n), so x is in E_n for some n.

$$E = \bigcup_{n=1}^{\infty} E_n$$

Now the events E_n are clearly mutually exclusive, so

$$P(E) = P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty}\left(\frac{26}{36}\right)^{n-1}\frac{4}{36}$$

$$\sum_{n=1}^{\infty}\left(\frac{26}{36}\right)^{n-1}\frac{4}{36} = \frac{4}{36}\,\frac{1}{1-\left(\frac{26}{36}\right)} = \frac{4}{10}$$

[recall geometric series: $\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r}$ if $|r|<1$]

32. b boys, g girls lined up in row
Each of $(b+g)!$ orderings is equally likely

Number girls by $j=1,2,\cdots,g$

Let E_j = event that person in ith spot is girl j

Then E_j consists of $(b+g-1)!$ outcomes (ordering of rest)

So $P(E_j) = \frac{(b+g-1)!}{(b+g)!} = \frac{1}{b+g}$

E = event that person in ith spot is a girl

$E = E_1 \cup E_2 \cup \cdots \cup E_g$.

The events E_j are mutually exclusive, so

$$P(E) = P(E_1 \cup \cdots \cup E_g) = \sum_{j=1}^{g} P(E_j) = \sum_{j=1}^{g} \frac{1}{b+g} = \frac{g}{b+g}$$

39. 5 hotels, 3 people check in.
P(each checks into different hotel) = ?

Assume that each person will choose between the 5 hotels independently, and is equally likely to go to any of the 5.

There are 5^3 outcomes, each with probability $\frac{1}{5^3}$
There are $5 \cdot 4 \cdot 3$ outcomes where no two choose same hotel,

so P(no two check into same hotel) $= \frac{5 \cdot 4 \cdot 3}{5^3}$

41. Die is rolled 4 times. Probability that 6 comes up at least once

Let E_i denote event that the i th roll is a 6. $(i=1,2,3,4)$

We want $P(E_1 \cup E_2 \cup E_3 \cup E_4)$

$= P(E_1) + P(E_2) + P(E_3) + P(E_4)$

$- P(E_1E_2) - P(E_1E_3) - P(E_1E_4) - P(E_2E_3) - P(E_2E_4) - P(E_1E_4)$

$+ P(E_1E_2E_3) + P(E_1E_2E_4) + P(E_1E_3E_4) + P(E_2E_3E_4)$

$- P(E_1E_2E_3E_4)$

$P(E_i) = \frac{1}{6}$ 4 terms like this

$P(E_iE_j) = \frac{1}{6^2}$ $6 = \binom{4}{2}$ terms like this

$P(E_iE_jE_k) = \frac{1}{6^3}$ $4 = \binom{4}{3}$ terms like this

$P(E_1E_2E_3E_4) = \frac{1}{6^4}$ $1 = \binom{4}{1}$ term like this

Total $= 4 \cdot \frac{1}{6} - 6 \cdot \frac{1}{6^2} + 4 \frac{1}{6^3} - \frac{1}{6^4} = P(E_1 \cup E_2 \cup E_3 \cup E_4)$

54. Probability that a bridge hand is void in at least one suit.

There are $\binom{52}{13}$ bridge hands, each equally likely.

V_1 = event that hand is void in spades
V_2 = hand is void in hearts
V_3 = hand is void in diamonds
V_4 = hand is void in clubs

We want $P(V_1 \cup V_2 \cup V_3 \cup V_4)$
use inclusion-exclusion

$$P(V_i) = \frac{\binom{39}{13}}{\binom{52}{13}}$$ since we must form hand out of 39 cards not of a particular suit.

$$P(V_i V_j) = \frac{\binom{26}{13}}{\binom{52}{13}}$$ can only use 26 cards not of suits i or j

$$P(V_i V_j V_k) = \frac{\binom{13}{13}}{\binom{52}{13}}$$ there is only one hand void in suits i, j, k; it consists of the fourth suit

$P(V_1 V_2 V_3 V_4) = 0$ No hand is void in all 4 suits

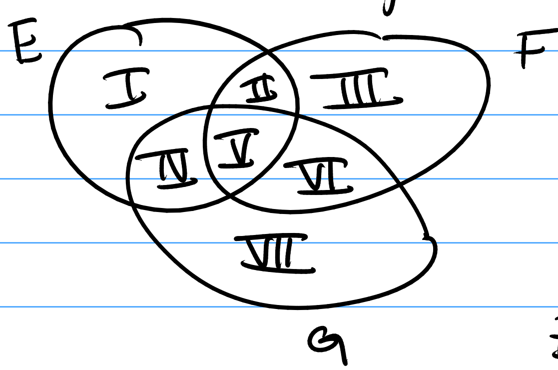
$$\text{so } P(V_1 \cup V_2 \cup V_3 \cup V_4) = \binom{4}{1}\frac{\binom{39}{13}}{\binom{52}{13}} - \binom{4}{2}\frac{\binom{26}{13}}{\binom{52}{13}} + \binom{4}{3}\frac{\binom{13}{13}}{\binom{52}{13}} - 0$$

$$= \left[4\binom{39}{13} - 6\binom{26}{13} + 4\right] \Big/ \binom{52}{13}$$

Ch 2 theoretical exercises.

10. Prove $P(E \cup F \cup G) = P(E) + P(F) + P(G)$
$$- P(E^cFG) - P(EF^cG) - P(EFG^c)$$
$$- 2P(EFG)$$

Break Venn diagram into 7 regions



$E = \mathrm{I} \cup \mathrm{II} \cup \mathrm{IV} \cup \mathrm{V}$
$F = \mathrm{II} \cup \mathrm{III} \cup \mathrm{V} \cup \mathrm{VI}$
$G = \mathrm{IV} \cup \mathrm{V} \cup \mathrm{VI} \cup \mathrm{VII}$

$E^cFG = \mathrm{VI}$
$EF^cG = \mathrm{IV}$ $\qquad EFG = \mathrm{V}$
$EFG^c = \mathrm{II}$

So $P(E) + P(F) + P(G) = P(\mathrm{I}) + \cancel{2}P(\mathrm{II}) + P(\mathrm{III}) + \cancel{2}P(\mathrm{IV}) + \cancel{3}P(\mathrm{V}) + \cancel{2}P(\mathrm{VI}) + P(\mathrm{VII})$

$-P(E^cFG) - P(EF^cG) - P(EFG^c) = -\cancel{P(\mathrm{VI})} - \cancel{P(\mathrm{IV})} - \cancel{P(\mathrm{II})}$

$-2P(EFG) \qquad = -\cancel{2P(\mathrm{V})}$

Add up $\qquad P(\mathrm{I}) + P(\mathrm{II}) + P(\mathrm{III}) + P(\mathrm{IV}) + P(\mathrm{V}) + P(\mathrm{VI}) + P(\mathrm{VII})$

the expression at the bottom equals $P(E \cup F \cup G)$
so we are done.

11. If $P(E) = .9$ and $P(F) = .8$ show $P(EF) \geq .7$
in general show
$$P(EF) \geq P(E) + P(F) - 1$$

Proof By inclusion-exclusion
$$P(E \cup F) = P(E) + P(F) - P(EF)$$

by axiom 1, $P(E \cup F) \leq 1$

so $1 \geq P(E) + P(F) - P(EF)$

$$P(EF) \geq P(E) + P(F) - 1$$

In the special case $P(EF) \geq .9 + .8 - 1 = .7$

18. f_n = # of way of tossing a coin n times so that no successive heads appear.

Clearly $f_1 = 2$ $f_2 = 3$

Consider f_n $n > 3$
if we get a tails first, there are f_{n-1} ways of doing the rest $(n-1)$ flips so that no successive heads appear
if we get a heads first, we need to get a tails second, and then there are f_{n-2} ways of doing the rest $(n-2)$ flips so that no successive

heads appear. Thus $f_n = f_{n-1} + f_{n-2}$

let P_n denote the probability of no successive heads when a coin is flipped n times.

$$P_n = \frac{f_n}{2^n}$$ since all 2^n outcomes are equally likely

What is P_{10}

$$
\left.\begin{aligned}
f_1 &= 2 \\
f_2 &= 3 \\
f_3 &= 2+3=5 \\
f_4 &= 3+5=8 \\
f_5 &= 5+8=13 \\
f_6 &= 8+13=21 \\
f_7 &= 13+21=34 \\
f_8 &= 21+34=55 \\
f_9 &= 34+55=89
\end{aligned}\right\} \text{Cf. Fibonacci numbers}
$$

$f_{10} = 55+89 = 144$ $\qquad P_{10} = 144/2^{10} = 144/1024$

19. An urn contains n red and m blue balls withdrawn one at a time until r $(r \leq n)$ red balls are gotten. find probability that k balls are withdrawn.

k balls are withdrawn if r-1 red balls are among the first k-1, and the kth ball is red

NOTE: This solution is using conditional probability

Prob. to choose $r-1$ red out of $k-1$ trys

$$= \frac{\binom{n}{r-1}\binom{m}{(k-1)-(r-1)}}{\binom{n+m}{k-1}} = \frac{\binom{n}{r-1}\binom{m}{k-r}}{\binom{n+m}{k-1}}$$

Prob. to choose kth ball to be red, once $r-1$ red, and $k-1$ total, balls have been removed:

$$= \frac{n-(r-1)}{n+m-(k-1)} = \frac{n-r+1}{n+m-k+1}$$

so over-all: $P(\text{k balls withdrawn}) = \frac{\binom{n}{r-1}\binom{m}{k-r}}{\binom{n+m}{k-1}} \cdot \frac{n-r+1}{n+m-k+1}$

Ch 3 theoretical exercises

3.2 Let $A \subset B$. Express as simply as possible

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)} \quad \left(AB = A \text{ since } A \subset B \right)$$

$$P(A|B^c) = \frac{P(AB^c)}{P(B^c)} = 0 \quad \left(\begin{array}{c} A \subset B \Rightarrow AB^c \subset BB^c = \emptyset \\ AB^c = \emptyset \end{array} \right)$$

$$P(B|A) = \frac{P(BA)}{P(A)} = \frac{P(A)}{P(A)} = 1 \quad \left(BA = A \text{ since } A \subset B \right)$$

$$P(B \mid A^c) = \frac{P(BA^c)}{P(A^c)}$$

$$= \frac{P(B) - P(A)}{1 - P(A)}$$

$B = BA^c \cup BA = BA^c \cup A$

so $P(B) = P(BA^c) + P(A)$

Hence $P(BA^c) = P(B) - P(A)$

Also $P(A^c) = 1 - P(A)$