Solutions 2

Ch 1 Theoretical exercises

13. Show, for $n>0$ $\sum_{i=6}^{n}(-1)^i\binom{n}{i}=0$

Use Binomial theorem $\sum_{i=0}^{n}\binom{n}{i}x^i y^{n-i}=(x+y)^n$

Plug in $x=-1$ and $y=1$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i(1)^{n-i}=(-1+1)^n=0^n=0$$

$(1)^{n-i}=1$ so get $\sum_{i=0}^{n}\binom{n}{i}(-1)^i=0$ QED

14. From set of n people committee of size j and a subcommittee of size i are chosen.

(a) choose committee : $\binom{n}{j}$

then choose subcommittee there of : $\binom{j}{i}$

so get $\binom{n}{j}\binom{j}{i}$

choose smaller subcommittee first $\binom{n}{i}$

then choose rest of committee $\binom{n-i}{j-i}$

so get $\binom{n}{i}\binom{n-i}{j-i}$

Thus $\binom{n}{j}\binom{j}{i}=\binom{n}{i}\binom{n-i}{j-i}$

(b) Prove $\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} = \binom{n}{i} 2^{n-i}$ $(i \leq n)$

Consider $\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} = \sum_{j=i}^{n} \binom{n}{i}\binom{n-i}{j-i}$ by part (a)

$= \binom{n}{i} \sum_{j=i}^{n} \binom{n-i}{j-i}$ since $\binom{n}{i}$ is independent of j

Now $\sum_{j=i}^{n} \binom{n-i}{j-i} = \binom{n-i}{0} + \binom{n-i}{1} + \cdots + \binom{n-i}{n-i}$

so $= \sum_{k=0}^{n-i} \binom{n-i}{k}$ (reindexing)

In general $\sum_{k=0}^{N} \binom{N}{k} = 2^N$

by binomial thm see p. 9, Example 4e

But $\sum_{k=0}^{n-i} \binom{n-i}{k} = 2^{n-i}$

So get $\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} = \binom{n}{i} 2^{n-i}$ QED.

(c) Prove $\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i}(-1)^{n-j} = 0$

$\sum_{j=1}^{n} \binom{n}{j}\binom{j}{i}(-1)^{n-j} = \sum_{j=i}^{n} \binom{n}{i}\binom{n-i}{j-i}(-1)^{n-j}$ by (a)

$= \binom{n}{i} \sum_{j=i}^{n} \binom{n-i}{j-i}(-1)^{n-j}$

So Need To Show $\sum_{j=i}^{n} \binom{n-i}{j-i} (-1)^{n-j} = 0$

this is true by exercise 13

set $N = n-i$, $K = j-i$, then $n-j = N-K$

so $\sum_{j=i}^{n} \binom{n-i}{j-i} (-1)^{n-j} = \sum_{K=0}^{N} \binom{N}{K} (-1)^{N-K}$

$= \sum_{K=0}^{N} \binom{N}{K} (-1)^{N-K} (1)^{K} = (1-1)^{N} = 0.$ QED.

Ch 2 problems

1. Box with 1 red, 1 green, 1 blue marble

Experiment I: draw marble, replace, draw again

S = all pairs of colors = { (R,R) (R,G) (R,B)
(G,R) (G,G) (G,B)
(B,R) (B,G) (B,B) }

$3^2 = 9$ points in S.

Experiment II: draw marble, don't replace, draw again

S = all pairs of distinct colors = { (R,G) (R,B)
(G,R) (G,B)
(B,R) (B,G) }

$3 \cdot 2 = 6$ points in S.

2. Die is rolled until a 6 appears, then stop

$S = \{$ any sequence of digits chosen from 1–5, and end on a 6 $\}$

e.g. 121351426 or 115436
or 1111 ⋯ (just 1's forever)
or 123123123⋯ forever.

E_n = event that experiment lasts n rolls
= $\{$ sequences of length n $\}$

e.g. 115436 is in E_6

$\bigcup_{n=1}^{\infty} E_n$ = event that experiment eventually ends

$\left(\bigcup_{n=1}^{\infty} E_n\right)^c$ = event that experiment never ends
= event that we never get a 6.

3. Two dice thrown E = sum of dice is odd
F = at least one die is 1
G = sum of dice is 5.

EF = sum is odd and at least one die is 1
$E \cup F$ = sum is odd or one die is 1 or both
FG = sum of dice is 5 and at least on is 1
= one die is 1 and the other is 4.
EF^c = sum is odd and neither die is 1.
EFG = sum is 5 and one die is 1 = FG (since 5 implies odd)
$G \subset E$

4. A, B, C take turns fliping a coin (A first, B second, C third)
first to get heads wins.

Sample space is $S = \begin{cases} 1, 01, 001, 0001, \dots \\ 000\cdots \end{cases}$

(a) Interpret sample space:

0 represents tails, 1 heads, sequence is
A's flip, B's flip, C's flip, A's flip, etc.

(b) Define in terms of S:

(i) A wins $= \{1, 0001, 0000001, \dots\}$

$= \{$3k 0's, followed by 1, for any $k = 0, 1, 2, \dots\}$

(ii) B wins $= \{01, 00001, 00000001, \dots\}$

$= \{$3k+1 0's, followed by 1, for any $k = 0, 1, \dots\}$

(iii) $(A \cup B)^c = A^c B^c$

$= \{001, 000001, \dots\} \cup \{000\cdots\}$

$= \{$3k+2 0's, followed by 1$\} \cup \{$infinitely many 0's$\}$

= C wins or game never ends.

5. System has 5 components

Let $(x_1, x_2, x_3, x_4, x_5)$ be a vector denoting whether each component is working or not

$0 \to$ not working
$1 \to$ working

So $(1,1,0,0,1)$ means 1, 2, 5 working
3, 4 not

(a) How many outcomes? each x_i either 0 or 1 $\to 2^5 = 32$

(b) W = (1 and 2 work) or (3 and 4 work) or (1, 3, 5 work)

Outcomes in W:

1 & 2 working	3 & 4	1 & 3 & 5
1 1 1 1 1	1 1 1 1 1	1 1 1 1 1
1 1 1 1 0	1 1 1 1 0	1 1 1 0 1
1 1 1 0 1	1 0 1 1 1	1 0 1 1 1
1 1 0 1 1	0 1 1 1 1	1 0 1 0 1
1 1 1 0 0	1 0 1 1 0	
1 1 0 1 0	0 1 1 1 0	
1 1 0 0 1	0 0 1 1 1	
1 1 0 0 0	0 0 1 1 0	

= Duplicate

15 outcomes

(c) A = 4 and 5 not working
outcomes in A = $2^3 = 8$ since other 3 parts can be in any state.

(d) Outcomes in AW: look at list for (b), see which end in 00:

11100 — 2 outcomes in AW.
11000

Ch 2 Theoretical exercises.

Prove the relation:

1. $EF \subset E \subset E \cup F$

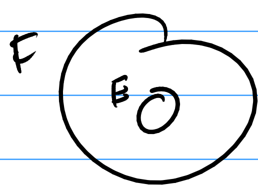
if E and F occur, then E occurs so $EF \subset E$

If E occurs, then E or F or both occur so $E \subset E \cup F$

(could also use Venn diagram)

2. If $E \subset F$, then $F^c \subset E^c$

If E occurs, then F occurs. So if F does not occur, then E cannot occur, so $F^c \subset E^c$.

Venn: 

outside F implies outside E

3. $F = FE \cup FE^c$ and $E \cup F = E \cup E^c F$

Proof: $F = FS = F(E \cup E^c) = FE \cup FE^c$ (distributive law)

QED

For second part, use $F = FE \cup FE^c$ and take union with E

$$E \cup F = E \cup (FE \cup FE^c) = (E \cup FE) \cup FE^c$$

Now, $E \cup FE = E$, since $FE \subset E$ (by exercise 1).

so $E \cup F = E \cup FE^c$ QED.

6. E, F, G three events

(a) only E $= EF^c G^c$

(b) both E & G, not F $= EGF^c$

(c) at least one $= E \cup F \cup G$

(d) at least two $= EF \cup EG \cup FG$

(e) all three $= EFG$

(f) none $= E^c F^c G^c = (E \cup F \cup G)^c$

(g) at most one $=$ (at least two)$^c = (EF \cup EG \cup FG)^c$
$=$ at least two of E^c, F^c, G^c $= E^cF^c \cup E^cG^c \cup F^cG^c$

(h) at most two $=$ (all three)$^c = (EFG)^c$
$= E^c \cup F^c \cup G^c$ $=$ at least one of E^c, F^c, G^c.

(i) exactly two $= EFG^c \cup EF^cG \cup E^cFG$

(j) At most 3 $= S$

7. Simplify:

(a) $(E \cup F)(E \cup F^c) = EE \cup EF \cup EF^c \cup FF^c$ (distributive law)

$= E \cup EF \cup EF^c \cup \emptyset = \boxed{E}$ since EF and EF^c are contained in E.

(b) $(E \cup F)(E^c \cup F)(E \cup F^c)$

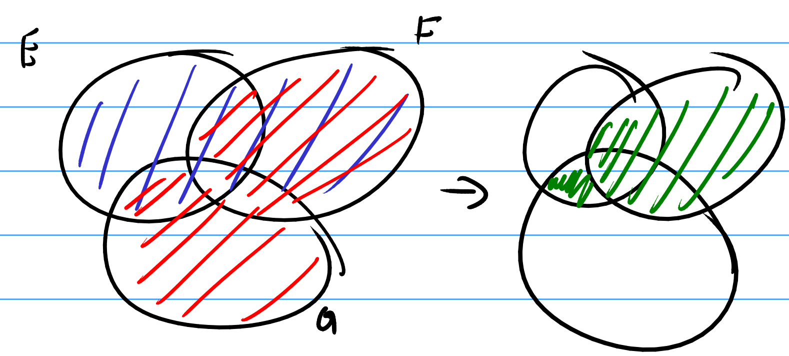
$= \underbrace{(E \cup F)(E \cup F^c)}(E^c \cup F)$ commutative law

$= E \quad (E^c \cup F)$ by (a)

$= EE^c \cup EF = \emptyset \cup EF = EF$

(c) $(E \cup F)(F \cup G) = EF \cup FF \cup EG \cup FG$ distributive

$= \underbrace{EF \cup FG}_{\text{these are contained in } F} \cup F \cup EG = F \cup EG$



9. Run experiment n times, get outcomes $x_1, \ldots, x_n$

for event E, define $n(E)$ number of times E occured in these trials $=$ # of x_i such that x_i is in E

Define $f(E) = \frac{n(E)}{n}$. We show it satisfies Axioms 1,2,3:

<u>Axiom 1</u>: By defintion $0 \leq n(E) \leq n$ so

$$0 \leq \frac{n(E)}{n} \leq 1$$

<u>Axiom 2</u>: Since S occured n times, $n(S) = n$

so $f(S) = 1$

<u>Axiom 3</u>: $E_1, E_2, E_3,$ mutually exclusive $E_iE_j = \emptyset$

$n\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} n(E_i)$ since each time E_i occurs in the sequence $x_1, \ldots, x_n$, the other E_j do not occur.

so $f\left(\bigcup_{i=1}^{\infty} E_i\right) = \frac{n\left(\bigcup_{i=1}^{\infty} E_i\right)}{n} = \sum_{i=1}^{\infty} \frac{n(E_i)}{n} = \sum_{i=1}^{\infty} f(E_i)$ ∎