Solutions 14

7.32 $X_i = \begin{cases} 1 & \text{if } i\text{th urn is empty} \\ 0 & \text{otherwise} \end{cases}$

$$E[X_i] = P\{i\text{th urn is empty}\} = \frac{i-1}{n}$$ from previous HW set.

$X = \sum_{i=1}^{n} X_i = \#$ of empty urns

$$Var(X) = Cov(X,X) = Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j\right)$$

$$= \sum_{i=1}^{n} Cov(X_i, X_i) + 2\sum_{i<j} Cov(X_i, X_j)$$

$$= \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)$$

$$Var(X_i) = E[X_i^2] - E[X_i]^2$$

$X_i^2 = X_i$ since X_i has values 0 or 1,
so $E[X_i^2] = \frac{i-1}{n}$

$$Var(X_i) = \frac{i-1}{n} - \left(\frac{i-1}{n}\right)^2 = \frac{i-1}{n}\left(1 - \frac{i-1}{n}\right)$$

$$Cov(X_i, X_j) = E[X_iX_j] - E[X_i]E[X_j]$$

$X_iX_j = \begin{cases} 1 & \text{if both urn } i \text{ and urn } j \text{ are empty} \\ 0 & \text{otherwise} \end{cases}$

$$E[X_iX_j] = P\{\text{both urn } i \text{ and urn } j \text{ are empty}\}$$

Assume $i < j$. Then all balls before ith cannot go into either, all balls from i to $j-1$ can go into i but not j, balls from j to n can go into either

$$P\{i \text{ and } j \text{ empty}\} = \underbrace{\frac{i-1}{\cancel{i}} \frac{\cancel{i}}{\cancel{i+1}} \cdots \frac{\cancel{j-2}}{j-1}}_{\text{simplifies to } \frac{i-1}{j-1}} \underbrace{\frac{j-2}{\cancel{j}} \frac{j-1}{j+1} \frac{\cancel{j}}{j+2} \cdots \frac{\cancel{n-3}}{n-1} \frac{\cancel{n-2}}{n}}_{\text{simplifies to } \frac{(j-2)(j-1)}{(n-1)\,n}}$$

$$= \frac{(i-1)(j-2)}{(n-1)\,n}$$

$$E[X_i] = \frac{i-1}{n} \qquad E[X_j] = \frac{j-1}{n}$$

$$\mathrm{Cov}(X_i, X_j) = \frac{(i-1)(j-2)}{(n-1)\,n} - \frac{i-1}{n}\frac{j-1}{n}$$

$$= \frac{i-1}{n}\left(\frac{j-2}{n-1} - \frac{j-1}{n}\right)$$

$$\text{So } \mathrm{Var}(X) = \sum_{i=1}^{n} \frac{i-1}{n}\left(1 - \frac{i-1}{n}\right) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} \frac{i-1}{n}\left(\frac{j-2}{n-1} - \frac{j-1}{n}\right)$$

7.36 $X = \#$ of 1's in n rolls of fair die
$Y = \#$ of 2's in n rolls of fair die

Let $X_i = \begin{cases} 1 & \text{if } i\text{th roll is } 1 \\ 0 & \text{otherwise} \end{cases} \qquad Y_j = \begin{cases} 1 & \text{if } j\text{th roll is } 2 \\ 0 & \text{otherwise} \end{cases}$

So $X = \sum_{i=1}^{n} X_i \qquad Y = \sum_{j=1}^{n} Y_j$

$$\mathrm{Cov}(X,Y) = \mathrm{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Cov}(X_i, Y_j)$$

if $i \neq j$, then X_i and Y_j are independent, so $\mathrm{Cov}(X_i, Y_j) = 0$.

if $i = j$, then X_i and Y_i are actually dependent

$$\mathrm{Cov}(X_i, Y_i) = E[X_i Y_i] - E[X_i]E[Y_i]$$

Now $E[X_i] = P\{i\text{th roll is } 1\} = 1/6$

$E[Y_i] = P\{i\text{th roll is } 2\} = 1/6$

Also $X_i Y_i = \begin{cases} 1 & \text{if } i\text{th roll is both a 1 and a 2} \\ 0 & \text{otherwise} \end{cases}$

So $X_i Y_i = 1$ is actually impossible!

$P\{X_i Y_i = 1\} = 0$, so $E[X_i Y_i] = 0$

Thus $\mathrm{Cov}(X_i, Y_i) = 0 - \frac{1}{6}\frac{1}{6} = -\frac{1}{36}$

So $\mathrm{Cov}(X,Y) = \sum_{i=1}^{n} \mathrm{Cov}(X_i, Y_i) = n\left(-\frac{1}{36}\right) = \frac{-n}{36}$

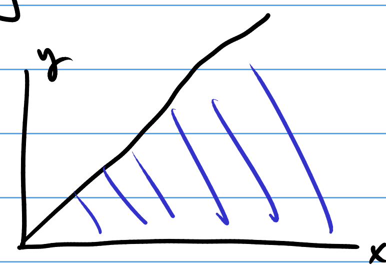
7.38 X and Y have joint density

$$f(x,y) = \begin{cases} 2e^{-2x}/x & 0 \le x < \infty,\ 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$E[X] = \iint x f(x,y)\,dx\,dy$$

Region



$$= \int_0^\infty \int_0^x x \cdot \frac{2e^{-2x}}{x}\,dy\,dx$$

$$= \int_0^\infty \int_y^\infty 2e^{-2x}\,dx\,dy = \int_0^\infty \left[-e^{-2x}\right]_y^\infty dy$$

$$= \int_0^\infty e^{-2y}\,dy = \left[-\tfrac{1}{2}e^{-2y}\right]_0^\infty = 0 - \left(-\tfrac{1}{2}\right)e^{-2\cdot 0} = \frac{1}{2}$$

$$E[Y] = \iint y f(x,y)\,dx\,dy = \int_0^\infty \int_0^x y \frac{2e^{-2x}}{x}\,dy\,dx$$

$$= \int_0^\infty \frac{2e^{-2x}}{x}\left[\frac{y^2}{2}\right]_0^x dx = \int_0^\infty \frac{x^2}{2}\,\frac{2e^{-2x}}{x}\,dx$$

$$= \int_0^\infty x e^{-2x}\,dx = \left[-\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}\right]_0^\infty$$

$$= -0 - 0 - \left(-0 - \tfrac{1}{4}\right) = \frac{1}{4}$$

$$E[XY] = \iint xy\, f(x,y)\,dx\,dy$$

$$= \int_0^\infty \int_0^x xy\, \frac{2e^{-2x}}{x}\, dy\, dx = \int_0^\infty \int_0^x y \cdot 2e^{-2x}\, dy\, dx$$

$$= \int_0^\infty \left[2e^{-2x} \frac{y^2}{2}\right]_{y=0}^{y=x} dx = \int_0^\infty x^2 e^{-2x}\, dx$$

$$= \left[-\frac{1}{4} e^{-2x}\left(1+2x+2x^2\right)\right]_0^\infty = 0 + \frac{1}{4} e^{-2\cdot 0}\left(1+2\cdot 0+2\cdot 0^2\right)\Big]$$

$$= \frac{1}{4}$$

$$Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{4} - \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$

Theoretical exercises

7.19 Assume X and Y are identically distributed but not necessarily independent.

$$Cov(X+Y, X-Y) = Cov(X,X) + Cov(X,-Y) + Cov(Y,X) + Cov(Y,-Y)$$

$$Cov(X,-Y) = -Cov(X,Y)$$
$$Cov(Y,X) = Cov(X,Y)$$
$$Cov(X,X) = Var(X)$$
$$Cov(Y,-Y) = -Cov(Y,Y) = -Var(Y)$$

$$Cov(X+Y, X-Y) = Var(X) - Var(Y) + Cov(X,Y) - Cov(X,Y)$$
$$= Var(X) - Var(Y)$$

Since X and Y have same distribution,

$Var(X) = Var(Y)$

so $Cov(X+Y, X-Y) = 0$

Ch. 8 problems:

8.1 Suppose X has mean 20 and variance 20

$P\{0<X<40\} = P\{|X-20|<20\}$

By Chebyshev, $P\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$

So $P\{|X-20| \geq 20\} \leq \frac{20}{(20)^2} = \frac{1}{20}$

so $P\{|X-20|<20\} \geq 1-\frac{1}{20} = \frac{19}{20}$

$P\{0<X<40\} \geq \frac{19}{20}$

8.4 $X_1, \dots, X_{20}$ independent Poisson random variables with mean 1.

So $X = \sum_{i=1}^{20} X_i$ is Poisson with parameter $\lambda = 20$ $\quad \mu = 20$, $\sigma^2 = 20$

(a) $P\{X>15\} \leq \frac{E[X]}{15} = \frac{20}{15} = \frac{4}{3}$ by Markov.

(b) By Central limit theorem, X is approximately normal, as it is the sum of many small independent increments.

$P\{X > 15\} = P\{X > 15.5\}$ (continuity correction, as X is discrete)

$\mu = 20 \quad \sigma = \sqrt{20}$

$$P\{X > 15.5\} = P\left\{Z > \frac{15.5 - 20}{\sqrt{20}}\right\} = P\{Z > -1.00623\}$$

$$= P\{Z < 1.00623\} = \Phi(1.00623) = .8428$$

(computer)

8.5 50 numbers rounded and then added.

Let X_i = rounding error on ith number.

$$X = \text{total error} = \sum_{i=1}^{50} X_i.$$

Errors are independent and uniformly distributed on $(-.5, .5)$

$$\mu = E[X_i] = \frac{.5 + -.5}{2} = 0$$

$$\sigma^2 = Var(X_i) = \frac{.5 - (-.5)}{12} = \frac{1}{12}$$

Central limit theorem $\Rightarrow$

$$\frac{\sum_{i=1}^{50} X - 50\mu}{\sqrt{50}\,\sigma} = \frac{X}{\sqrt{50}\sqrt{1/12}}$$ is approximately standard normal

$$P\{|X|>3\} = P\left\{\left|\frac{X}{\sqrt{50/12}}\right| > \frac{3}{\sqrt{50/12}}\right\}$$

$$= P\{|Z| > 1.46969\} = 2(1-\Phi(1.46969))$$

$$= 0.1416 \quad \text{(computer)}$$

8.7 100 identical light bulbs

X_i = lifetime of ith bulb is exponential with $\mu = 5$ hours

$$5 = \mu = \frac{1}{\lambda} \Rightarrow \lambda = 1/5$$

$$\sigma^2 = Var(X_i) = \frac{1}{\lambda^2} = 25, \quad \sigma = 5$$

$X = \sum_{i=1}^{100} X_i$ is total lifetime of bulbs

By CLT: $\dfrac{X - 100\mu}{\sigma\sqrt{100}} = \dfrac{X - 500}{50}$ is approximately standard normal

$$P\{X > 525\} = P\left\{\frac{X-500}{50} > \frac{525-500}{50}\right\} = P\left\{Z > \frac{1}{2}\right\}$$

$$= 1 - P\{Z < \tfrac{1}{2}\} = 1 - \Phi(\tfrac{1}{2}) = .3085$$

Theoretical exercise

8.1 Show if X has mean μ and standard deviation σ

Then $P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$

Proof from Markov: $E[(X-\mu)^2] = \sigma^2$

$$P\{(X-\mu)^2 \geq k^2\sigma^2\} \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2}$$

But $(X-\mu)^2 \geq k^2\sigma^2 \iff |X-\mu| \geq k\sigma$

so $P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$

Proof from Chebyshev:

Chebyshev says $P\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$

set $l = \frac{k}{\sigma}$ so $k = \sigma l$

then we get $P\{|X-\mu| \geq \sigma l\} \leq \frac{\sigma^2}{(\sigma l)^2} = \frac{1}{l^2}$

Change dummy variable l back to k to get desired form.