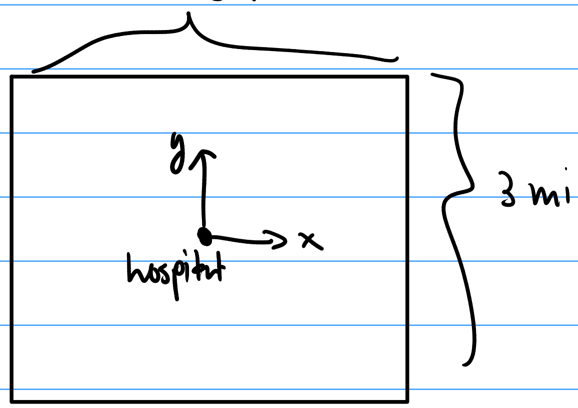
Solutions 13

Problems

7.5



- Hospital at $(0,0)$
- Accident at (X,Y) uniformly in the square
  $-1.5 < X < 1.5$
  $-1.5 < Y < 1.5$
- Due to rectangular street grid, travel distance is $|X|+|Y|$

Expected distance

$$E\big[|X|+|Y|\big] = \iint (|x|+|y|)\, f(x,y)\,dx\,dy$$

Now, $f(x,y) = \begin{cases} \frac{1}{9} & \text{if } \begin{array}{l} -1.5<x<1.5 \\ -1.5<y<1.5 \end{array} \\ 0 & \text{otherwise} \end{cases}$

so $$E\big[|X|+|Y|\big] = \int_{-1.5}^{1.5}\int_{-1.5}^{1.5} (|x|+|y|)\frac{1}{9}\,dx\,dy$$

Break into 4 parts depending on signs of x and y:

$x>0, y>0$: $$\int_0^{1.5}\int_0^{1.5}(x+y)\frac{1}{9}\,dx\,dy = \frac{1}{9}\int_0^{1.5}\left[\frac{x^2}{2}+xy\right]_0^{1.5} dy$$

$$= \frac{1}{9}\int_0^{1.5}\left(\frac{(1.5)^2}{2}+1.5y\right)dy = \frac{1}{9}\left[\frac{1.5^2}{2}y + 1.5\frac{y^2}{2}\right]_0^{1.5} = \frac{1}{9}\left[\frac{(1.5)^3}{2}+\frac{(1.5)^3}{2}\right]$$

$$= \frac{1}{9}(1.5)^3 = \frac{1}{9}\cdot\frac{3^3}{2^3} = \frac{3}{8}$$

By symmetry, the other 3 regions
- $x > 0, y < 0$
- $x < 0, y > 0$
- $x < 0, y < 0$

all give the same value $\frac{3}{8}$

so $E\big[|X| + |Y|\big] = 4 \cdot \frac{3}{8} = \boxed{\frac{3}{2}}$

7.9 Balls $1, \dots, n$ placed in urns $1, \dots, n$ in such a way that ball i has equal chance of being in urns $1, \dots, i$.

(a) Expected # of urns that are empty $= X$

$X = X_1 + X_2 + \cdots + X_n$

where $X_i = \begin{cases} 1 & \text{if urn } i \text{ is empty} \\ 0 & \text{otherwise} \end{cases}$

$E[X_i] = 0 P\{X_i = 0\} + 1 \cdot P\{X_i = 1\} = P\{X_i = 1\}$

Need $P\{X_i = 1\} = P\{\text{urn } i \text{ is empty}\}$

$= P\{\text{ball } i \text{ not in urn } i\} P\{\text{ball } i+1 \text{ not in urn } i\} \cdots P\{\text{ball } n \text{ not in urn } i\}$

$= \frac{i-1}{i} \cdot \frac{i}{i+1} \cdot \frac{i+1}{i+2} \cdot \cdots \cdot \frac{n-1}{n} = \frac{i-1}{n}$

So $E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{(i-1)}{n} = \frac{1}{n}\left[\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right] = \frac{1}{n}\left[\frac{(n+1)n}{2} - n\right]$

$= \frac{n+1}{2} - 1 = \boxed{\frac{n-1}{2}}$

(b) P{ none of the urns are empty }

No urn empty ⇒ ball n in urn n ⇒ ball n-1 in urn n-1
⇒ ball n-2 in urn n-2 ⇒ ... ⇒ ball 1 in urn 1

so Probability $= \frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{1}{n-2} \cdots \frac{1}{1} = \boxed{\frac{1}{n!}}$

7.11 n idependent flips of a coin (P(heads) = p)

X = # changeovers (... HT ... or ... TH ...)

Let $X_i = \begin{cases} 1 & \text{if change over between } i \text{ and } i+1 \\ 0 & \text{otherwise} \end{cases}$

$E[X_i] = P\{X_i = 1\} = P(HT \text{ or } TH) = p(1-p) + (1-p)p = 2p(1-p)$

For n flips we have (n-1) such variables, as there are n-1 places where a changeover can happen

$X = \sum_{i=1}^{n-1} X_i$

$E[X] = \sum_{i=1}^{n-1} E[X_i] = \sum_{i=1}^{n-1} 2p(1-p) = \boxed{2(n-1)p(1-p)}$

Theoretical Exercises

7.4 X has finite mean μ and variance σ^2.

$g(x) = g(\mu) + g'(\mu)(x-\mu) + \frac{g''(\mu)}{2}(x-\mu)^2 + \cdots$

$$E[g(X)] = E\left[g(\mu) + g'(\mu)(X-\mu) + \frac{g''(\mu)}{2}(X-\mu)^2 + \cdots\right]$$

$$= g(\mu) + g'(\mu)\, E[X-\mu] + \frac{g''(\mu)}{2} E[(X-\mu)^2] + \cdots$$

$$= g(\mu) + \frac{g''(\mu)}{2}\sigma^2$$

since $E[X-\mu] = E[X] - \mu = \mu - \mu = 0$

and $E[(X-\mu)^2] = 0$ by definition.

7.5 $A_1, \ldots, A_n$ events

C_k = event that at least k A_i's occur.

Let $X = \#\, A_i$'s that occur $\quad X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$

Then $X = \sum_{i=1}^{n} X_i$

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} P\{X_i = 1\} = \sum_{i=1}^{n} P(A_i)$$

On the other hand: $C_k = P\{X \geq k\}$

$$\sum_{k=1}^{n} P(C_k) = \sum_{k=1}^{n} P\{X \geq k\} = \sum_{k=1}^{n} \sum_{j=k}^{n} P\{X = j\}$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{j} P\{X = j\}$$ sum over pairs $k \leq j$ reverse order of summation

$$= \sum_{j=1}^{n} j \cdot P\{X = j\} = E[X]$$ by definition

Hence $\sum_{k=1}^{n} P(C_k) = E[X] = \sum_{i=1}^{n} P(A_i)$