Solutions 12

Problems

6.18 Two points on Line : X uniform over $(0, \frac{L}{2})$
Y uniform over $(\frac{L}{2}, L)$

X and Y are independent

Distance between $= Y - X$ (since Y is to right of X)

$P\{Y - X > \frac{L}{3}\}$?

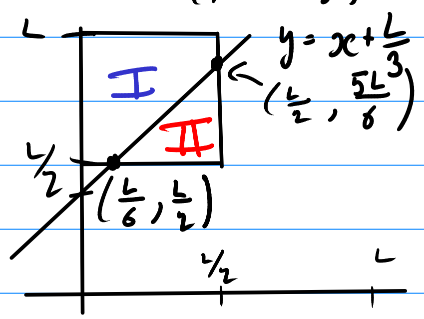
Density function : $f_X(x) = \begin{cases} 2/L & 0 < x < L/2 \\ 0 & \text{otherwise} \end{cases}$

$$f_Y(y) = \begin{cases} 2/L & L/2 < y < L \\ 0 & \text{otherwise} \end{cases}$$

Joint density : $f(x,y) = \begin{cases} \frac{4}{L^2} & 0 < x < \frac{L}{2} \text{ and } \frac{L}{2} < y < L \\ 0 & \text{otherwise} \end{cases}$

$$P\{Y - X > \tfrac{L}{3}\} = \iint_{\{y-x > \frac{L}{3}\}} f(x,y)\,dx\,dy = \iint_{\substack{y-x > \frac{L}{3} \\ 0 < x < \frac{L}{2} \\ \frac{L}{2} < y < L}} \frac{4}{L^2}\,dx\,dy$$

Region



$$= \frac{4}{L^2} \text{Area}(\mathrm{I})$$

$$= 1 - \frac{4}{L^2} \text{Area}(\mathrm{II})$$

$$\text{Area}(\mathrm{II}) = \frac{1}{2}\left(\frac{L}{3}\right)^2 = \frac{1}{2}\frac{L^2}{9}$$

$$P\{Y-X > \tfrac{L}{3}\} = 1 - \frac{4}{L^2}\left(\frac{1}{2}\frac{L^2}{9}\right) = 1 - \frac{2}{9} = \frac{7}{9}$$

6.20 joint density $f(x,y) = \begin{cases} xe^{-(x+y)} & x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$

X and Y are independent

$x>0$: $f_X(x) = \int_0^\infty xe^{-(x+y)}dy = xe^{-x}\int_0^\infty e^{-y}dy = xe^{-x}[0+1] = xe^{-x}$

$x<0$ $f_X(x) = 0$

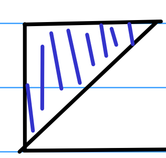
$y>0$ $f_Y(y) = \int_0^\infty xe^{-(x+y)}dx = e^{-y}\int_0^\infty xe^{-x}dx = e^{-y}\left[-xe^{-x} - e^{-x}\right]_{x=0}^{\infty}$

$= e^{-y}[-0-0+0+1] = e^{-y}$

$y<0$ $f_Y(y) = 0.$

So $f(x,y) = f_X(x)\cdot f_Y(y)$, and the variables are independent

In the case $f(x,y) = \begin{cases} 2 & 0<x<y,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$



Well: $f_X(x) = \begin{cases} \int_x^1 2dy & \text{if } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

$f_X(x) = \begin{cases} 2(1-x) & \text{if } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

$$f_Y(y) = \begin{cases} \int_0^y 2\,dx & \text{if } 0<y<1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2y & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore f_X(x) f_Y(y) = \begin{cases} 4(1-x)y & \text{if } 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

and $f_X(x) f_Y(y) \neq f(x,y)$

So X and Y are not independent.

6.29 X = weekly sales is normal w/ $\mu = 2200$ $\sigma = 230$

(a) $P\{$total sales over two weeks $> 5000\}$

let X_1 = sales in first week, X_2 = sales in second week $\}$ each normal $\mu = 2200$ $\sigma = 230$

Assuming X_1 and X_2 are independent:

$X_1 + X_2$ is normal with $\mu = 2200 + 2200 = 4400$

$$\sigma = \sqrt{(230)^2 + (230)^2} \approx 325.3$$

$$P\{X_1 + X_2 > 5000\} = P\left\{Z_1 > \frac{5000-4400}{325.3}\right\} \approx P\{Z > 1.84\}$$

$$= 1 - P\{Z < 1.84\} = 1 - \Phi(1.84) \approx 1 - .97 = .03$$

(b) $P\{$sales > 2000 in atleast 2 out of 3 weeks$\}$

X_1, X_2, X_3 sales in the 3 weeks.

For each week: $P\{X_i > 2000\} = P\{Z > \frac{2000-2200}{230}\} \approx P\{Z > -.87\}$

$= P\{Z < .87\} = \Phi(.87) \approx .808$

P{at least 2 out of 3 are >2000}

$= P\{X_1 > 2000, X_2 > 2000, X_3 \leq 2000\}$

$+ P\{X_1 > 2000, X_2 \leq 2000, X_3 > 2000\}$

$+ P\{X_1 \leq 2000, X_2 > 2000, X_3 > 2000\}$

$+ P\{X_1 > 2000, X_2 > 2000, X_3 > 2000\}$

$= 3(.808)^2(1-.808) + (.808)^3$ { assuming independence.

$= 3 \cdot (.653) \cdot (.192) + (.528) = .904$

6.30 X = Jill's score is normal w/ $\mu = 170$ $\sigma = 20$
Y = Jack's score is normal w/ $\mu = 160$ $\sigma = 15$

Assume X and Y independent.

P{Jack's score is higher} $= P\{Y > X\} = P\{X - Y < 0\}$

Y normal w/ $\mu = 160$ $\sigma = 15$ $\Rightarrow$ $-Y$ normal w/ $\mu = -160$
$\sigma = 15$

$\Rightarrow$ $X - Y = X + (-Y)$ is normal w/ $\mu = 170 + (-160) = 10$
$\sigma = \sqrt{20^2 + 15^2} = 25$

$$P\{X-Y<0\} \underbrace{=}_{\text{continuity correction}} P\{X-Y<-.5\} = P\{Z<\frac{-.5-10}{25}\} = P\{Z<-.42\}$$

$$= P\{Z>.42\} = 1-P\{Z<.42\} = 1-\Phi(.42) = 1-.6628$$

$$= .3372$$

(b) $P\{X+Y>350\}$?

$X+Y$ is normal w/ $\mu=170+160=330$ $\quad \sigma=\sqrt{20^2+15^2}=25$

continuity correction

so $P\{X+Y>350\} = P\{X+Y>350.5\} = P\{Z>\frac{350.5-330}{25}\} = P\{Z>.82\}$

$$= 1-\Phi(.82) \approx 1-.7939 = .2061$$

6.32 X = # typos on page has expected value .2

** We will model X as a Poisson RV. with $\lambda=.2$

Article has 10 pages : X_i = # typos on ith page.

** Assume pages are independent ; so all X_i are independent

Sum of Poisson RV-s is poisson

so X_1+X_2 Poisson with $\lambda=.2+.2=.4$

$\underbrace{X_1+X_2+\cdots+X_9+X_{10}}_{10}$ is Poisson with $\lambda=\underbrace{.2+\cdots+.2}_{10}=2$

$$P\left\{\sum_{i=1}^{10} X_i = 0\right\} = e^{-2}$$

$$P\left\{\sum_{i=1}^{10} X_i \geq 2\right\} = 1 - P\left\{\sum_{i=1}^{10} X_i = 0\right\} - P\left\{\sum_{i=1}^{10} X_i = 1\right\}$$

$$= 1 - e^{-2} - e^{-2}\frac{2^1}{1!} = 1 - 3e^{-2}$$

6.38 Choose $X \in \{1,2,3,4,5\}$ then choose $Y \leq X$

(a) joint PMF

X \ Y	1	2	3	4	5	P_X
1	1/5	0	0	0	0	1/5
2	1/10	1/10	0	0	0	1/5
3	1/15	1/15	1/15	0	0	1/5
4	1/20	1/20	1/20	1/20	0	1/5
5	1/25	1/25	1/25	1/25	1/25	1/5
P_Y	.457	.257	.157	.09	.04	

probability may not add to 1 due to rounding

(b) Conditional mass function of X

$p(x|y)$

x \ y	1	2	3	4	5
1	.438	0	0	0	0
2	.219	.389	0	0	0
3	.146	.259	.425	0	0
4	.109	.195	.318	.556	0
5	.088	.156	.255	.444	1

(c) Are X and Y independent: NO, $p(x|y)$ depends very much on the value of y.

6.42 Joint density is given by

$$f(x,y) = c(x^2-y^2)e^{-x} \quad \text{for } 0 \le x < \infty \quad -x \le y \le x$$

$$f_X(x) = \int_{-x}^{x} c(x^2-y^2)e^{-x}\,dy = ce^{-x}\int_{-x}^{x}(x^2-y^2)\,dy$$

$$= ce^{-x}\left[x^2y - \frac{y^3}{3}\right]_{y=-x}^{y=x} = ce^{-x}\left[x^3 - \frac{x^3}{3} - x^2(-x) + \frac{(-x)^3}{3}\right]$$

$$= ce^{-x}\left[x^3 - \frac{x^3}{3} + x^3 - \frac{x^3}{3}\right] = ce^{-x}\cdot\frac{4}{3}x^3$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{c(x^2-y^2)e^{-x}}{ce^{-x}\cdot\frac{4}{3}\cdot x^3} = \frac{3}{4}\left(\frac{1}{x} - \frac{y^2}{x^3}\right)$$

This is for $-x \le y \le x$ $\quad f(y|x) = 0$ otherwise

Distribution $F_{Y|X}(a|x) = \int_{-\infty}^{a} f(y|x)\,dy$

$$= \int_{-x}^{a} \frac{3}{4}\left(\frac{1}{x} - \frac{y^2}{x^3}\right)dy = \frac{3}{4}\left[\frac{y}{x} - \frac{y^3}{3x^3}\right]_{y=-x}^{y=a}$$

$$= \frac{3}{4}\left[\frac{a}{x} - \frac{a^3}{3x^3} - \frac{-x}{x} + \frac{(-x)^3}{3x^3}\right] = \frac{3}{4}\left[\frac{a}{x} - \frac{a^3}{3x^3} + 1 - \frac{1}{3}\right]$$

$$F(y|x) = \frac{3a}{4x} - \frac{a^3}{4x^3} + \frac{1}{2} \quad (\text{valid for } -x \le a \le x)$$

$$F(y|x) = 0 \quad \text{if } a < -x$$

$$F(y|x) = 1 \quad \text{if } a > x$$

Theoretical exercises

6.11 X_1, X_2, X_3, X_4, X_5 be independent continuous R.V.s having common distribution F and density f

$$I = P\{X_1 < X_2 < X_3 < X_4 < X_5\}$$

(a) $$I = \underset{\{x_1<x_2<x_3<x_4<x_5\}}{\iiiint\!\!\int} f(x_1) f(x_2) f(x_3) f(x_4) f(x_5)\, dx_1\, dx_2\, dx_3\, dx_4\, dx_5$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{x_5} \int_{-\infty}^{x_4} \int_{-\infty}^{x_3} \int_{-\infty}^{x_2} f(x_1) \cdots f(x_5)\, dx_1\, dx_2\, dx_3\, dx_4\, dx_5$$

substitute:

$u_i = F(x_i) \qquad du_i = f(x_i)\, dx_i$

$x_i = -\infty \iff u_i = 0 \qquad x_5 = \infty \iff u_5 = 1$

$x_i < x_{i+1} \iff u_i < u_{i+1}$

$$I = \int_0^1 \int_0^{u_5} \int_0^{u_4} \int_0^{u_3} \int_0^{u_2} du_1\, du_2\, du_3\, du_4\, du_5$$

so I does not depend on F or f!

(b) $\int_0^{u_2} du_1 = u_2$, $\int_0^{u_3} u_2\, du_2 = \frac{u_3^2}{2}$, $\int_0^{u_4} \frac{u_3^2}{2} du_3 = \frac{u_4^3}{6}$

$\int_0^{u_5} \frac{u_4^3}{6} du_4 = \frac{u_5^4}{24}$, $\int_0^1 \frac{u_5^4}{24} = \frac{1}{120} = \frac{1}{5!}$

(c) Intuitive explanation: since $X_1, \dots, X_5$ all have same distribution, each of the $5!$ numerical orders of the values is equally likely to occur.

since $X_1 < X_2 < X_3 < X_4 < X_5$ is one of these orders, it's probability is $\frac{1}{5!}$

6.14 X and Y independent geometric random variables with same parameter p.

(a) $P\{X=i \mid X+Y=n\}$? In Bernoulli trials w/ parameter p, X is # of trials to first success, and X+Y is number of trials to second success. So given that second success occurs on nth trial, there are (n-1) possible trials where the first success could have occurred. It seems that any of these (n-1) possibilities is equally likely, so we conjecture

$$P\{X=i \mid X+Y=n\} = \frac{1}{n-1}$$

(b) Proof: $P\{X=i \mid X+Y=n\} = \frac{P\{X=i, X+Y=n\}}{P\{X+Y=n\}} = \frac{P\{X=i, Y=n-i\}}{P\{X+Y=n\}}$

$= \frac{P\{X=i\}P\{n-i\}}{P\{X+Y=n\}}$ since X and Y are independent.

$P\{X=i\} = (1-p)^{i-1}p \qquad P\{Y=n-i\} = (1-p)^{n-i-1}p$

$$P\{X+Y=n\} = \sum_{i=1}^{n-1} P\{X=i\}P\{Y=n-i\} = \sum_{i=1}^{n-1} (1-p)^{i-1}p(1-p)^{n-i-1}p$$

$$= \sum_{i=1}^{n-1} (1-p)^{n-2}p^2 = (n-1)(1-p)^{n-2}p^2$$

[Could also say $X+Y$ is negative binomial w/ p, $r=2$]

$$P\{X=i \mid X+Y=n\} = \frac{(1-p)^{i-1}p\,(1-p)^{n-i-1}p}{(n-1)(1-p)^{n-2}p^2} = \frac{1}{n-1} \quad \checkmark$$