# Solutions 11

## Problems

5.32 $X$ = time in hours to repair machine is exponential with $\lambda = \frac{1}{2}$

(a) $P(X>2) = 1 - F(2) = 1 - \left[1 - e^{-\lambda \cdot 2}\right] = e^{-\lambda \cdot 2} = e^{-\frac{1}{2}\cdot 2} = e^{-1}$

(b) $P(X>10 \mid X>9) = P(X>1)$ by memorylessness

$= e^{-\lambda \cdot 1} = e^{-(\frac{1}{2})}$

5.34 $X$ = # miles (in thousands) car can be driven before complete breakdown.

Assume $X$ is exponential with $\lambda = \frac{1}{20}$

Assume car has 10,000 miles already, what is probability it last another 20,000?

$P(X>30 \mid X>10) = P(X>20)$ by memorylessness

$= e^{-\lambda(20)} = e^{-\frac{1}{20}20} = e^{-1}$

Now assume $X$ is uniformly distributed in the interval $(0,40)$

$$P(X>30 \mid X>10) = \frac{P(X>30)}{P(X>10)} = \frac{(10/40)}{(30/40)} = \frac{10}{30} = \frac{1}{3}$$

Since: $P(X>30) = \frac{40-30}{40-0} = \frac{10}{40} = \frac{1}{4}$ $\quad P(X>10) = \frac{40-10}{40-0} = \frac{30}{40} = \frac{3}{4}$

5.39 X is exponential with parameter $\lambda = 1$

$$f_X = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \le 0 \end{cases} = \begin{cases} e^{-x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

$Y = \log X$. find density function of $Y$

here $g(x) = \log x$ $\quad g^{-1}(y) = e^y$

$(g^{-1})'(y) = e^y$

So $f_Y(y) = f_X(g^{-1}(y)) \cdot (g^{-1})'(y) = f_X(e^y) \cdot e^y$

$= e^{-e^y} \cdot e^y = e^{y - e^y}$

Theoretical exercises

5.13 Median is $m$ such that $F(m) = \frac{1}{2}$

(a) X is uniform over $(a, b)$

$$F_X = \begin{cases} 0 & x \le a \\ \frac{x-a}{b-a} & a < x \le b \\ 1 & b < x \end{cases}$$

so $F(m) = \frac{1}{2} \Rightarrow \frac{m-a}{b-a} = \frac{1}{2} \Rightarrow m - a = \frac{1}{2}(b-a) \Rightarrow m = \frac{1}{2}(b+a)$

$\therefore$ m = median = midpoint of $(a, b)$

(b) $X$ normal with mean $\mu$ and variance $\sigma^2$

$$F_X(x) = P\{X \le x\} = P\{Z \le \frac{x-\mu}{\sigma}\} = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

standard normal

$$\frac{1}{2} = F_X(m) = \Phi\left(\frac{m-\mu}{\sigma}\right) \Rightarrow \frac{m-\mu}{\sigma} = \Phi^{-1}\left(\frac{1}{2}\right)$$

Now $\Phi(0) = \frac{1}{2}$, so $\Phi^{-1}\left(\frac{1}{2}\right) = 0$

$\Rightarrow \frac{m-\mu}{\sigma} = 0 \Rightarrow m = \mu$ so median = mean for normal random variable

(c) $X$ is exponential with rate $\lambda$.

$$F_X(x) = \begin{cases} 1-e^{-\lambda x} & x>0 \\ 0 & x \le 0 \end{cases}$$

$$\frac{1}{2} = F_X(m) = 1 - e^{-\lambda m} \Rightarrow e^{-\lambda m} = \frac{1}{2} \Rightarrow -\lambda m = \log \frac{1}{2} = -\log 2$$

$\Rightarrow m = \frac{\log 2}{\lambda}$.

Note: Expectation $= \frac{1}{\lambda}$, so median $\neq$ mean for exponential dist.

5.30 $X$ has density function $f_X$. Let $Y = aX+b$
Find $f_Y$, the density function of $Y$

Note: I will assume $a > 0$, for in this case $g(x) = ax+b$ is an increasing function, which was the only situation we discussed in class. If $a<0$ there is a similar solution, while if $a=0$ then $Y$ does not have a density function, as it is constant, and hence discarde.

Assume $a>0$, then $g(x)=ax+b$ is an increasing functi

$$g^{-1}(y) = \frac{x-b}{a} \qquad (g^{-1})'(y) = \frac{1}{a}$$

$$f_Y(y) = f_X(g^{-1}(y))\,(g^{-1})'(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

If $a<0$, then $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

If $a=0$, then $Y=b$ is a discrete R.V. with probability mass functi $p(y)=\begin{cases} 1 & y=b \\ 0 & y\neq b \end{cases}$

5.31 $X$ normal R.V. with mean $\mu$ and variance $\sigma^2$

$Y=e^X \qquad g(x)=e^x \qquad$ range $=(0,\infty)$

$$g^{-1}(y) = \log y \qquad (g^{-1})'(y) = \frac{1}{y}$$

So for $y\in(0,\infty)$

$$f_X = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

$$f_Y(y) = f_X(\log y)\,\frac{1}{y}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma}\cdot\frac{1}{y}\cdot\exp\left\{-\frac{(\log y-\mu)^2}{2\sigma^2}\right\} \quad \text{for } y>0$$

$$f_Y(y) = 0 \quad \text{for } y\leq 0$$

# Ch 6 problems

6.1 2 fair dice are rolled: find joint probability mass function if:

(a) $X$ = max value, $Y$ = sum of values

| $X \backslash Y$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1/36 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 2/36 | 1/36 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3 | 0 | 0 | 2/36 | 2/36 | 1/36 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 2/36 | 2/36 | 2/36 | 1/36 | 0 | 0 | 0 | 0 |
| 5 | 0 | 0 | 0 | 0 | 2/36 | 2/36 | 2/36 | 2/36 | 1/36 | 0 | 0 |
| 6 | 0 | 0 | 0 | 0 | 0 | 2/36 | 2/36 | 2/36 | 2/36 | 2/36 | 1/36 |

(b) $X$ = first die, $Y$ = max value

| $X \backslash Y$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 2 | 0 | 2/36 | 1/36 | 1/36 | 1/36 | 1/36 |
| 3 | 0 | 0 | 3/36 | 1/36 | 1/36 | 1/36 |
| 4 | 0 | 0 | 0 | 4/36 | 1/36 | 1/36 |
| 5 | 0 | 0 | 0 | 0 | 5/36 | 1/36 |
| 6 | 0 | 0 | 0 | 0 | 0 | 6/36 |

(c) $X$ = min value
$Y$ = max value

| $X \backslash Y$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1/36 | 2/36 | 2/36 | 2/36 | 2/36 | 2/36 |
| 2 | 0 | 1/36 | 2/36 | 2/36 | 2/36 | 2/36 |
| 3 | 0 | 0 | 1/36 | 2/36 | 2/36 | 2/36 |
| 4 | 0 | 0 | 0 | 1/36 | 2/36 | 2/36 |
| 5 | 0 | 0 | 0 | 0 | 1/36 | 2/36 |
| 6 | 0 | 0 | 0 | 0 | 0 | 1/36 |

6.7 $X_1$ = # failures before first success
$X_2$ = # failures between first and second successes.

$X_1$ and $X_2$ are discrete R.V.s with possible values
$n, m = 0, 1, 2, \ldots$

Joint PMF: $p(n,m) = P\{X_1 = n, X_2 = m\}$

$$= P(\underbrace{FF\cdots F}_{n} S \underbrace{FF\cdots F}_{m} S) = (1-p)^n p (1-p)^m p$$

So $p(n,m) = (1-p)^{n+m} p^2$

6.9 Joint density is

$$f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \quad \text{for } 0<x<1, 0<y<2$$

(a) Verify normalization:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^2\int_0^1 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dx\,dy$$

$$= \frac{6}{7}\int_0^2 \left[\frac{x^3}{3} + \frac{x^2 y}{4}\right]_{x=0}^{x=1} dy = \frac{6}{7}\int_0^2 \left[\frac{1}{3} + \frac{y}{4}\right] dy$$

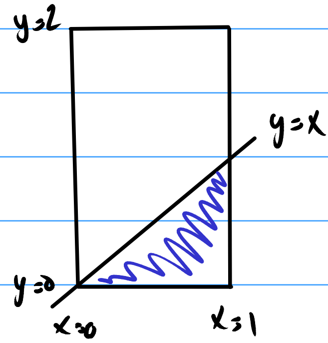
$$= \frac{6}{7}\left[\frac{y}{3} + \frac{y^2}{8}\right]_{y=0}^{y=2} = \frac{6}{7}\left[\frac{2}{3} + \frac{4}{8}\right] = \frac{6}{7}\left[\frac{2}{3} + \frac{1}{2}\right] = \frac{6}{7}\left[\frac{7}{6}\right] = 1. \checkmark$$

(b) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy$

$$= \frac{6}{7}\left[x^2 y + \frac{xy^2}{4}\right]_{y=0}^{y=2} = \frac{6}{7}\left[2x^2 + x\right]$$

(c) $P\{X>Y\} = \iint_{\{x>y\}} f(x,y)\,dx\,dy = \iint_{\substack{0<x<1 \\ 0<y<2 \\ x<y}} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dx\,dy$

Picture of region




$$= \int_0^1 \int_0^x \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy\,dx$$

$$= \frac{6}{7}\int_0^1 \left[x^2 y + \frac{xy^2}{4}\right]_{y=0}^{y=x} dx = \frac{6}{7}\int_0^1 \left[x^3 + \frac{x^3}{4}\right] dx$$

$$= \frac{6}{7}\cdot\frac{5}{4}\cdot\int_0^1 x^3\,dx = \frac{6}{7}\cdot\frac{5}{4}\cdot\left[\frac{x^4}{4}\right]_{x=0}^{x=1} = \frac{\cancel{6}^{3}}{7}\cdot\frac{5}{\cancel{4}_{2}}\cdot\frac{1}{4} = \frac{15}{56}$$

(d) $P\left(Y>\frac{1}{2} \mid X<\frac{1}{2}\right) = \dfrac{P\left(Y>\frac{1}{2} \text{ and } X<\frac{1}{2}\right)}{P\left(X<\frac{1}{2}\right)}$

$$P\left(X<\tfrac{1}{2}\right) = \int_0^{1/2} f_X(x)\,dx = \int_0^{1/2} \frac{6}{7}\left[2x^2 + x\right] dx$$

$$= \frac{6}{7}\left[\frac{2}{3}x^3 + \frac{x^2}{2}\right]_{x=0}^{x=1/2} = \frac{6}{7}\left[\frac{2}{3}\cdot\frac{1}{8} + \frac{1}{8}\right] = \frac{\cancel{6}^{3}}{7}\cdot\frac{5}{3}\cdot\frac{1}{\cancel{8}_{4}} = \frac{5}{28}$$

$$P\left(Y>\frac{1}{2}, X<\frac{1}{2}\right) = \int_{1/2}^{2}\int_{0}^{1/2} \frac{6}{7}\left(x^2+\frac{xy}{2}\right) dx\, dy$$

$$= \frac{6}{7}\int_{1/2}^{2}\left[\frac{x^3}{3}+\frac{x^2 y}{4}\right]_{x=0}^{x=1/2} dy = \frac{6}{7}\int_{1/2}^{2}\left[\frac{1}{24}+\frac{y}{16}\right] dy$$

$$= \frac{6}{7}\left[\frac{y}{24}+\frac{y^2}{32}\right]_{1/2}^{2} = \frac{6}{7}\left[\frac{1}{12}+\frac{1}{8}-\frac{1}{48}-\frac{1}{128}\right] = \frac{69}{448}$$

$$P\left(Y>\frac{1}{2} \middle| X<\frac{1}{2}\right) = \frac{69/448}{5/28} = \frac{69}{80}$$

(e) $$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 x \cdot \frac{6}{7} \cdot (2x^2+x)\, dx$$

$$= \int_0^1 \frac{6}{7}(2x^3+x^2)\, dx = \frac{6}{7}\left[\frac{x^4}{2}+\frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{6}{7}\left[\frac{1}{2}+\frac{1}{3}\right] = \frac{5}{7}$$

(f) $$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_0^1 \frac{6}{7}\left(x^2+\frac{xy}{2}\right) dx$$

$$= \frac{6}{7}\left[\frac{x^3}{3}+\frac{x^2 y}{4}\right]_{x=0}^{x=1} = \frac{6}{7}\left[\frac{1}{3}+\frac{y}{4}\right]$$

$$E[Y] = \int_0^2 y \cdot \frac{6}{7} \cdot \left[\frac{1}{3}+\frac{y}{4}\right] = \frac{6}{7}\int_0^2 \left(\frac{y}{3}+\frac{y^2}{4}\right) dy$$

$$= \frac{6}{7}\left[\frac{y^2}{6} + \frac{y^3}{12}\right]_0^2 = \frac{6}{7}\left[\frac{4}{6} + \frac{8}{12}\right] = \frac{8}{7}$$