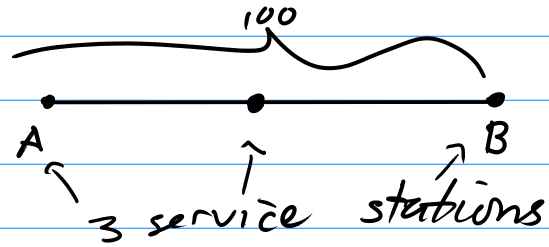
Solutions 10

5.12



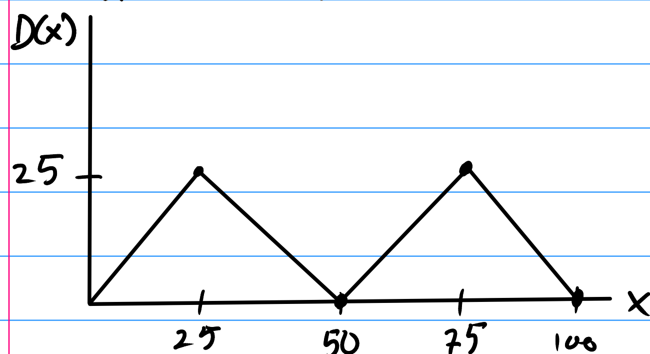
X = location of breakdown is uniform on $(0, 100)$

PDF $f(x) = \begin{cases} \frac{1}{100} & 0 < x < 100 \\ 0 & \text{otherwise} \end{cases}$

Let $D(X)$ be the distance from location of breakdown

$$D(X) = \begin{cases} X, & 0 < X \le 25 \\ 50 - X, & 25 < X \le 50 \\ X - 50, & 50 < X \le 75 \\ 100 - X, & 75 < X < 100 \end{cases}$$

Which looks like



$$E[D(X)] = \int_0^{100} D(x) f(x)\,dx = \int_0^{100} D(x) \frac{1}{100}\,dx = \frac{1}{100} \int_0^{100} D(x)\,dx$$

$= \frac{1}{100}$ (Area under the graph of $D(x)$)

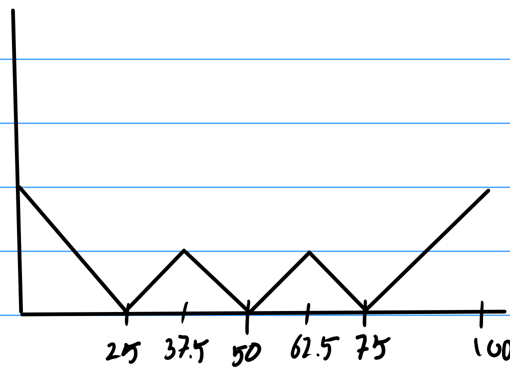
$= \frac{1}{100}(4)\left(\frac{1}{2}\, 25 \cdot 25\right)$ (break up into 4 triangles)

$= \frac{1}{2} \cdot 25 = 12.5$

If stations are located at 25, 50, and 75,

$$D(x) = \begin{cases} 25-x & 0<x\le 25 \\ x-25 & 25<x\le 37.5 \\ 50-x & 37.5<x\le 50 \\ x-50 & 50<x\le 62.5 \\ 75-x & 62.5<x\le 75 \\ x-75 & 75<x<100 \end{cases}$$

which looks like



$$E[D(X)] = \int_0^{100} D(x)\frac{1}{100}\,dx$$

$$= \frac{1}{100}(\text{Area under curve})$$

$$= \frac{1}{100}\left[4\cdot\left(\tfrac{1}{2}(12.5)^2\right) + 2\left(\tfrac{1}{2}\,25^2\right)\right]$$

$$= \frac{1}{100}[312.5 + 625] = \frac{1}{100}[937.5] = 9.375$$

Since $9.375 < 12.5$, the second scheme is more efficient.

5.13 You arrive a bus stop at 10.
Arrival of bus is uniformly distributed between 10 and 10:30.

(a) What is probability we have to wait longer than 10 minutes

Wait longer than 10 minutes $\iff$ bus arrives between 10:10 and 10:30

$$P(10{:}10 < X < 10{:}30) = \frac{20 \text{ minutes}}{30 \text{ minutes}} = \frac{2}{3}$$

(b) If, at 10:15 bus has not arrived, find probability you must wait <u>another</u> 10 minutes

Wait another 10 minutes $\Leftrightarrow$ bus arrives between 10:25 and 10:30

$$P(10{:}25 < X < 10{:}30 \mid 10{:}15 < X) = \frac{P(10{:}25 < X < 10{:}30)}{P(10{:}15 < X)}$$

$$\frac{5/30}{15/30} = \frac{5}{15} = \frac{1}{3}.$$

5.15 X normal with $\mu = 10$ and $\sigma^2 = 36$
compute $\sigma = 6$
let Z denote standard normal RV.

NOTE: I USED THE TABLE ON P. 201 to look up $\Phi(x)$
CALCULATOR MAY GIVE SLIGHTLY DIFFERENT RESULTS

(a) $P\{X > 5\} = P\left\{\frac{X-10}{6} > \frac{5-10}{6}\right\} = P\left\{Z > \frac{-5}{6}\right\} = P\left\{Z < \frac{5}{6}\right\}$

$$= \Phi\left(\tfrac{5}{6}\right) \approx \Phi(.83) \approx .7967$$

(b) $P\{4 < X < 16\} = P\left\{\frac{4-10}{6} < Z < \frac{16-10}{6}\right\} = P\{-1 < Z < 1\}$

$$= \Phi(1) - \Phi(-1) = \Phi(1) - [1 - \Phi(1)] = 2\Phi(1) - 1$$

$$= 2(.8413) - 1 = .6826$$

(c) $P\{X < 8\} = P\left\{Z < \frac{8-10}{6}\right\} = P\left\{Z < -\frac{1}{3}\right\} = 1 - \Phi\left(\frac{1}{3}\right)$

$$= 1 - \Phi(.33) = 1 - (.6293) = .3707$$

(d) $P\{X<20\} = P\{Z < \frac{20-10}{6}\} = P\{Z < \frac{5}{3}\}$

$= \Phi(\frac{5}{3}) \approx \Phi(1.66) = .9515$

(e) $P\{X>16\} = P\{Z > \frac{16-10}{6}\} = P\{Z>1\} = 1-\Phi(1)$

$= 1-.8413 = .1587$

5.16 X = annual rainfall is normal with $\mu = 40$, $\sigma = 4$
(X is in inches)

$P\{X>50\} = P\{\frac{X-40}{4} > \frac{50-40}{4}\} = P\{\frac{X-40}{4} > 2.5\}$

$= 1-\Phi(2.5) = 1-(.9938) = .0062$

The number of years it takes to have a year with over 50 inches of rain is a geometric random variable with $p = .0062$ (Assuming the rainfalls in different years are independent)

So $P\{$take over 10 years to get > 50 inches$\}$

$= (1-p)^{10} = (.9938)^{10} = .9397$

5.18 X is normal with $\mu = 5$ and unknown σ.
If we know $P\{X>9\} = .2$, then

$.2 = P\{\frac{X-5}{\sigma} > \frac{9-5}{\sigma}\} = P\{\frac{X-5}{\sigma} > \frac{4}{\sigma}\} = 1-\Phi(\frac{4}{\sigma})$

$\therefore \Phi(\frac{4}{\sigma}) = 1-.2 = .8$

Doing a reverse lookup in the table for Φ, we see

$\frac{4}{\sigma} \approx .85 \Rightarrow \sigma \approx \frac{4}{.85} = 4.7$

Thus $Var(X) = \sigma^2 \approx 22$

5.23 1000 independent rolls of a fair die
$X = \#$ of 6's rolled

Compute approximation to $P\{150 \le X \le 200\}$

X is binomial with $n = 1000$ $p = \frac{1}{6}$
$\mu = np = 166.6$
$\sigma^2 = np(1-p) = 138.8$ $\sigma = 11.79$

Apply DeMoivre-Laplace Limit theorem

$P\{150 \le X \le 200\} = P\{149.5 < X < 200.5\}$ continuity correction

$\approx P\left\{\frac{149.5 - 166.6}{11.79} < Z < \frac{200.5 - 166.6}{11.79}\right\}$

$\approx P\{-1.450 < Z < 2.875\}$

$= \Phi(2.875) - \Phi(-1.450)$

$\approx .9244$ Note: I used computer for some of these steps. Rounding and using table may give slightly different answer.

Second part: Assume 6 appears exactly 200 times

Find conditional probability that 5 appears less than 150 times.

FACT: Assuming 6 appears exactly 200 times, the number of 5's which appear is a binomial Random var. with $n=800$ and $p=\frac{1}{5}$

Quick intuitive argument: If 6 appears 200 times, we can set those 200 trials aside, and we have $n=800$ trials on which a 5 might appear. Since we know a 6 cannot appear on any of these 800 trials, the probability of getting a 5 is

$$P(5 \mid \text{not } 6) = \frac{P(5)}{P(\text{not } 6)} = \frac{1/6}{5/6} = \frac{1}{5}, \text{ so } p = \frac{1}{5}$$

Proof with formulas let $Y = \#$ of 5's $X = \#$ 6's

$$P(Y=k \mid X=200) = \frac{P(Y=k \,\&\, X=200)}{P(X=200)}$$

$$P(X=200) = \binom{1000}{200}\left(\frac{1}{6}\right)^{200}\left(\frac{5}{6}\right)^{800}$$

$$P(Y=k \,\&\, X=200) = \underbrace{\binom{1000}{200}}_{\text{where 6's go}}\underbrace{\binom{800}{k}}_{\text{where 5's go}}\underbrace{\left(\frac{1}{6}\right)^{200}}_{200\text{ 6's}}\underbrace{\left(\frac{1}{6}\right)^{k}}_{k\text{ 5's}}\underbrace{\left(\frac{4}{6}\right)^{800-k}}_{800-k\text{ 1,2,3,4's}}$$

$$P(Y=k \mid X=200) = \frac{\cancel{\binom{1000}{200}}\binom{800}{k}\cancel{\left(\frac{1}{6}\right)^{200}}\left(\frac{1}{6}\right)^{k}\left(\frac{4}{6}\right)^{800-k}}{\cancel{\binom{1000}{200}}\cancel{\left(\frac{1}{6}\right)^{200}}\left(\frac{5}{6}\right)^{800}}$$

$$= \binom{800}{k}\left(\frac{1}{6}\right)^{k}\left(\frac{4}{6}\right)^{800-k}\left(\frac{6}{5}\right)^{800} = \binom{800}{k} 1^{k} 4^{800-k} \cancel{\left(\frac{1}{6}\right)^{800}} \frac{\cancel{6^{800}}}{5^{800}}$$

$$= \binom{800}{k}\left(\frac{1}{5}\right)^{k}\left(\frac{4}{5}\right)^{800-k}$$

which is binomial with $n=800$ $p=\frac{1}{5}$

$P\{Y<150 \mid X=200\}$ can be approximated

$np = 160$ $np(1-p) = 128$ $\sqrt{np(1-p)} = 8\sqrt{2} \approx 11.31$

$$P\{Y<150 \mid X=200\} = P\{Y<149.5 \mid X=200\}$$

$$= P\left\{Z < \frac{149.5-160}{11.31}\right\}$$

$$= P\{Z < -.9284\} = \Phi(-.9284) \approx .1766$$

5.27 In 10000 tosses coin lands on heads 5800 times

Is it reasonable to conclude coin is unfair

Suppose coin is fair: then $X = \#$ heads is binomial with $n = 10000$, $p=\frac{1}{2}$, $\mu = np = 5000$, $\sigma^2 = np(1-p) = 2500$

$\sigma = \sqrt{2500} = 50$

since $\mu = 5000$, $\sigma = 50$, $5800 = \mu + 16\sigma$

So 5800 heads represents at deviation from mean of 16 standard deviations

So $P\{X \geq 5800\} = P\{Z \geq 16\} = 1 - \Phi(16)$, and this number is essentially zero.

In fact even:

$P\{X \geq 5100\} = P\{Z \geq 2\} \approx .02275 = 2.275\%$ is quite small

$P\{X \geq 5400\} = P\{Z \geq 8\} \approx 6.66 \times 10^{-16}$ is very very very small.

$P\{X \geq 5800\} = P\{Z \geq 16\}$ is so small it is outside the limits of precision on my computer.

So it is nearly certain the coin is <u>not</u> fair.

P. 227 – 228

5.2 Statement: $E[Y] = \int_0^\infty P\{Y > y\}\,dy - \int_0^\infty P\{Y < -y\}\,dy$

<u>Step 1</u>: $\int_0^\infty P\{Y > y\}\,dy = \int_0^\infty x f_Y(x)\,dx$

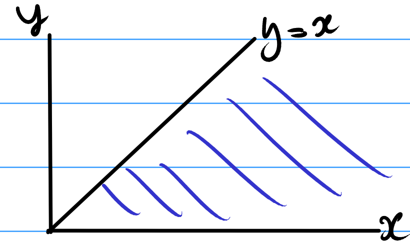
(this is the same proof as in class)

$$P\{Y > y\} = \int_y^\infty f_Y(x)\,dx$$

So $\int_0^\infty P\{Y>y\}dy = \int_0^\infty \int_y^\infty f_Y(x)\,dx\,dy$

the integral is over the region $\{(x,y): 0<y<\infty,\ y<x<\infty\}$

Picture:



$= \{(x,y): 0<x<\infty,\ 0<y<x\}$

Change order of integration: $= \int_0^\infty \int_0^x f_Y(x)\,dy\,dx$

$= \int_0^\infty \left[f_Y(x)\,y\right]_{y=0}^{y=x} dx = \int_0^\infty x f_Y(x)\,dx$ Done with Step 1.

Step 2 $\int_0^\infty P\{Y<-y\}dy = \int_{-\infty}^0 x f_Y(x)\,dx$

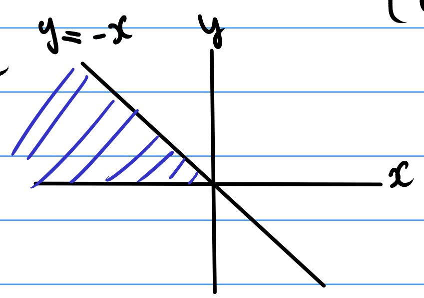
$P\{Y<-y\} = \int_{-\infty}^{-y} f_Y(x)\,dx$

So $\int_0^\infty P\{Y<-y\}dy = \int_0^\infty \int_{-\infty}^{-y} f_Y(x)\,dx\,dy$

Region of integration $\{(x,y): 0<y<\infty,\ -\infty<x<-y\}$

$= \{(x,y): -\infty<x<0,\ 0<y<-x\}$

Picture



Switch order of integration

$$= \int_{-\infty}^{0} \int_{0}^{-x} f_Y(x)\, dy\, dx = \int_{-\infty}^{0} \left[y f_Y(x) \right]_{y=0}^{y=-x} dx$$

$$= \int_{-\infty}^{0} (-x) f_Y(x) dx = -\int_{-\infty}^{0} f_Y(x) dx$$ Done with Step 2

Finally,

Step 3: $E[Y] = \int_{-\infty}^{\infty} x f_Y(x) dx$ by definition

$$= \int_{-\infty}^{0} x f_Y(x) dx + \int_{0}^{\infty} x f_Y(x) dx$$

$$= -\int_{0}^{\infty} P\{Y < -y\} dy + \int_{0}^{\infty} P\{Y > y\} dy$$ QED

5.3 Statement: Let $Y = g(X)$ then

$$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Using 5.2, we have

$$E[Y] = \int_{0}^{\infty} P\{Y > y\} dy - \int_{0}^{\infty} P\{Y < -y\} dy$$

$$= \int_{0}^{\infty} P\{g(X) > y\} dy - \int_{0}^{\infty} P\{g(X) < -y\} dy$$

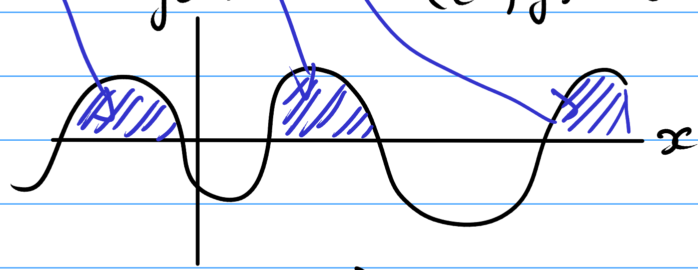
Analyze $\int_0^\infty P\{g(X) > y\}\,dy$

$$\left(P\{g(X) > y\} = \int_{\substack{x \text{ such} \\ \text{that } g(x) > y}} f_X(x)\,dx \right)$$

$$= \int_0^\infty \int_{\substack{x \text{ such that} \\ g(x) > y}} f_X(x)\,dx\,dy$$

Region of integration: $\{(x,y) : 0 < y < \infty,\ y < g(x)\}$

$= \{(x,y) : 0 < g(x),\ 0 < y < g(x)\}$



$$= \int_{\substack{x \text{ such} \\ \text{that } g(x) > 0}} \int_0^{g(x)} f_X(x)\,dy\,dx$$

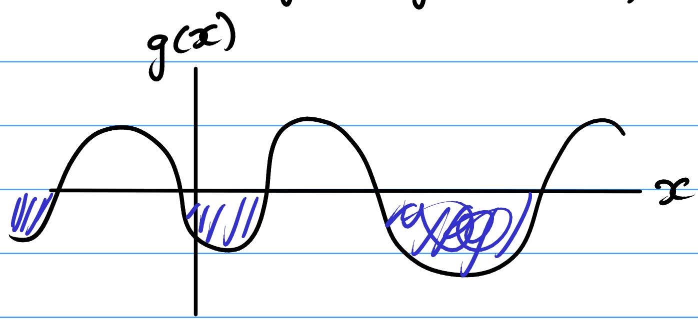
$$= \int_{\substack{x \text{ such that} \\ g(x) > 0}} g(x) f_X(x)\,dx$$

Analyze $-\int_0^\infty P\{g(X) < -y\}\,dy$

$$= -\int_0^\infty \int_{\substack{x \text{ such that} \\ g(x) < -y}} f_X(x)\,dx\,dy$$

Region : $\{(x,y) : 0 < y < \infty, \quad g(x) < -y\}$

$\{(x,y) : g(x) < 0, \quad 0 < y < -g(x)\}$



$$= -\int_{\substack{x \text{ such that} \\ g(x) < 0}} \int_0^{-g(x)} f_X(x)\,dx = -\int_{\substack{x \text{ such that} \\ g(x) < 0}} -g(x) f_X(x)\,dx$$

$$= \int_{\substack{x \text{ such that} \\ g(x) < 0}} g(x) f_X(x)\,dx$$

$$E[g(X)] = \int_{\substack{x \text{ such that} \\ g(x) > 0}} g(x) f_X(x)\,dx + \int_{\substack{x \text{ such that} \\ g(x) < 0}} g(x) f_X(x)\,dx$$

Now $\int_{\substack{x \text{ such that} \\ g(x) = 0}} g(x) f_X(x)\,dx = 0$

So we can neglect the set where $g(x)=0$, and we get

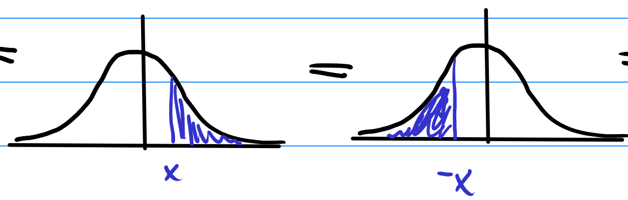
$$E[g(X)] = \int_{\text{all } x} g(x) f_X(x)\,dx = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

$\big(g(x)>0, g(x)<0, \text{or } g(x)=0 \big)$ QED.

5.9 Z standard normal random variable, and $x>0$

(a) $P\{Z>x\} = P\{Z<-x\}$

In pictures

$P\{Z>x\} =$ 

$= P\{Z<-x\}$

With integrals

$$P\{Z>x\} = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-y^2/2}\,dy$$

$u=-y$
$du=-dy$

$$= \frac{1}{\sqrt{2\pi}} \int_{-x}^{-\infty} e^{-u^2/2}(-du)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} e^{-u^2/2}\,du = P\{Z<-x\}$$

(b) $P\{|Z|>x\} = 2P\{Z>x\}$

Proof $|Z|>x$ iff $Z>x$ or $-Z>x$
iff $Z>x$ or $Z<-x$

$$P\{|Z|>x\} = P\{Z>x\}+P\{Z<-x\}$$
$$= P\{Z>x\}+P\{Z>x\} \quad \text{by part (a)}$$
$$= 2P\{Z>x\} \quad \text{QED}$$

(c) $P\{|Z|<x\} = 2P\{Z<x\}-1$

Proof: $P\{|Z|<x\} = 1 - P\{|Z|>x\}$
$$= 1-2P\{Z>x\} \quad \text{by part (b)}$$
$$= 1-2\left[1-P\{Z<x\}\right]$$
$$= 1-2+2P\{Z<x\}$$
$$= 2P\{Z<x\}-1 \quad \text{QED}$$