Solutions 1

Problems

1. (a) 7-place license plate: 2 letters, 5 numbers

$\# = 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$

(b) no letters or numbers repeated

$\# = 26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$

3. 20 workers, 20 jobs, one to each

$\#\text{ of ways} = \underbrace{20}_{\substack{\text{which}\\\text{job first}\\\text{worker does}}} \cdot \underbrace{19}_{\substack{\text{which}\\\text{job}\\\text{second}\\\text{worker}\\\text{does}}} \cdot 18 \cdot 17 \cdot \cdots 3 \cdot 2 \cdot 1 = 20!$

etc

5. Area codes: 3 digits

first digit 2 − 9
second digit 0 or 1
third digit 1 − 9

$\#\text{ area codes} = \underset{\substack{\text{first}\\\text{digit}}}{8} \cdot \underset{\substack{\text{second}\\\text{digit}}}{2} \cdot \underset{\substack{\text{third}\\\text{digit}}}{9}$

$\#\text{ area codes starting with } 4 = \underset{\substack{\text{first}\\\text{digit}}}{1} \cdot \underset{\substack{\text{second}\\\text{digit}}}{2} \cdot \underset{\substack{\text{third}\\\text{digit}}}{9}$

8. How many arrangements

(a) Fluke all letters different $\rightarrow 5!$

(b) Propose 2 Ps, 2 Os $\rightarrow \dfrac{7!}{2!2!}$

(c) Mississippi 4 Is, 4 Ss, 2 Ps $\rightarrow \dfrac{11!}{4!4!2!}$

(d) Arrange 2 As, 2 Rs $\rightarrow \dfrac{7!}{2!2!}$

9. 12 blocks, 6 black, 4 red, 1 white, 1 blue

$$\# \text{ arrangements} = \frac{12!}{6!\,4!\,1!\,1!} = \frac{12!}{6!\,4!}$$

11. Arranging books 3 novels, 2 math, 1 chem

(a) Books in any order $\rightarrow 6!$

(b) math books together
novels together
(chem together since only 1)

$$\rightarrow \underset{\substack{\text{order}\\\text{of}\\\text{subjects}}}{3!} \cdot \underset{\substack{\text{order}\\\text{of}\\\text{novels}}}{3!} \cdot \underset{\substack{\text{order}\\\text{of}\\\text{math}}}{2!} \cdot \underset{\text{chem}}{1!}$$

(c) Novels together, others in any order

Think of novels as one block; we're arranging the other 3 books and this one block

$\rightarrow$ 4! arrangements

The block of novels can be arranged in 3! ways

Total $= 4! \cdot 3!$

13. 20 people: every one shakes hands with everyone

handshakes $= \binom{20}{2}$ since each handshake involves an unordered set of 2 people

28. 8 teachers divided between 4 schools.

There are 4 choices for where each teacher can go.

$$\underbrace{4}_{\text{first teacher}} \cdot \underbrace{4}_{\text{second teacher}} \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^8$$

, etc.

What if each school recieves 2 teachers?
First school get 2 of 8, second gets 2 of the six that remain, so # of way is

$$\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{8!}{6!\,2!}\,\frac{6!}{4!\,2!}\,\frac{4!}{2!\,2!} = \frac{8!}{(2!)^4}$$

Theoretical Exercises

2. Two experments: first has m outcomes
   # of outcomes of second experiment depends on the outcome of first experiment:

   if first experment has outcome i, then second experiment has n_i outcomes

   Table of outcomes of the pair:

   $(1,1)$ $(1,2)$ $\cdots$ $(1,n_1)$

   $(2,1), (2,2)$ $\cdots\cdots$ $(2,n_2)$

   $\vdots$

   $(m,1),(m,2) \cdots (m,n_m)$

   n_i outcomes in ith row:

   $$\text{total} = n_1+n_2+\cdots+n_m = \sum_{i=1}^{m} n_i$$

3. Select r objects from n; order of selection is relevent:

   $$\underset{\substack{\text{first}\\\text{object}\\\text{chosen}}}{n} \cdot \underset{\substack{\text{second}\\\text{object}\\\text{chosen}}}{(n-1)} \cdot \underset{\text{3rd}}{(n-2)} \cdots \underset{r\text{th}}{(n-(r-1))}$$

   when choosing rth object $(r-1)$ are gone

Don't divide by $r!$ because order does matter!

4. There are $\binom{n}{r}$ arrangements of n balls of which r are black and $n-r$ are white

Prove this combinatorially:

When arranging the balls, we have n places to put them in. Once we choose which places contain black balls, the placement of the white balls is determined (the white balls go in the places left over)

$$\text{so } \#(\text{arrangements}) = \#\begin{pmatrix}\text{choose } r \text{ places out of } n \\ (\text{where the black balls go})\end{pmatrix}$$

$$= n \text{ choose } r = \binom{n}{r}$$

8. Prove $\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}$

Proof: consider a group of $n+m$ people, n men
m women

$$\# \text{ groups of } r \text{ people} = \binom{n+m}{r}$$

On the other hand, we can seperate out the groups that have all women, 1 men and $r-1$ women, 2 men and $r-2$ women, etc.

groups with

0 men, r women $\rightarrow \binom{n}{0}\binom{m}{r}$

1 men, r-1 women $\rightarrow \binom{n}{1}\binom{m}{r-1}$

2 men, r-2 women $\binom{n}{2}\binom{m}{r-2}$

$\vdots$

r-1 men, 1 women $\binom{n}{r-1}\binom{m}{1}$

r men, 0 women $\binom{n}{r}\binom{m}{0}$

Adding these up gives all possible $\binom{n+m}{r}$ groups

So: $\binom{n}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}$

10. Group of n people: choose committee of size k, and a chairperson for this committee.

Count possibilities 3 ways

(a) committee first, then chair

$$\left.\begin{array}{l}\#\text{ committees } = \binom{n}{k} \\ \#\text{ possible chairs, given committee} = k\end{array}\right\} \rightarrow \binom{n}{k}\cdot k$$

(b) choose committee minus chair, then chair

\# committees minus chair = # groups of k-1 out of n

$$= \binom{n}{k-1}$$

\# chairs, given that we've chosen k-1 non-chairs

$$= (n-(k-1)) = (n-k+1)$$

$$\Rightarrow \binom{n}{k-1}(n-k+1)$$

(c) choose chair, then choose rest of committee

\#chairs $= n$

\# groups of k-1 for rest of committee, given one person is already chosen to be chair

$$= \binom{n-1}{k-1}$$

$$\Rightarrow n\binom{n-1}{k-1}$$

(d) since we counted the same thing 3 ways, they must be equal

$$\binom{n}{k}k = \binom{n}{k-1}(n-k+1) = n\binom{n-1}{k-1}$$

(c) To verify this using the formula:

$$k\binom{n}{k} = k\frac{n!}{(n-k)!\,k!} = \cancel{k}\frac{n!}{(n-k)!\,(\cancel{k}\cdot(k-1)\cdots 2\cdot 1)}$$

$$= \frac{n!}{(n-k)!\,(k-1)!}$$

$$\binom{n}{k-1}(n-k+1) = \frac{n!}{(n-(k-1))!\,(k-1)!}(n-k+1)$$

$$= \frac{n!}{\cancel{(n-k+1)}(n-k)!\,(k-1)!}\cancel{(n-k+1)}$$

$$= \frac{n!}{(n-k)!\,(k-1)!}$$

$$n\binom{n-1}{k-1} = n\frac{(n-1)!}{[(n-1)-(k-1)]!\,(k-1)!} = n\frac{(n-1)!}{(n-k)!\,(k-1)!}$$

$$= \frac{n!}{(n-k)!\,(k-1)!}$$

So all three equal $\frac{n!}{(n-k)!\,(k-1)!}$ ✓