Solutions 1

Problems

1. (a) 7-place license plate: 2 letters, 5 numbers

$$\# = 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$$

(b) no letters or numbers repeated

$$\# = 26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$$

3. 20 workers, 20 jobs, one to each

$$\#\text{ of ways} = \underbrace{20}_{\substack{\text{which}\\\text{job first}\\\text{worker does}}} \cdot \underbrace{19}_{\substack{\text{which}\\\text{job}\\\text{second}\\\text{worker}\\\text{does}}} \cdot 18 \cdot 17 \cdots 3 \cdot 2 \cdot 1 = 20!$$

etc

5. Area codes: 3 digits

first digit 2 – 9
second digit 0 or 1
third digit 1 – 9

$$\#\text{ area codes} = \underset{\substack{\text{first}\\\text{digit}}}{8} \cdot \underset{\substack{\text{second}\\\text{digit}}}{2} \cdot \underset{\substack{\text{third}\\\text{digit}}}{9}$$

$$\#\text{ area codes starting with } 4 = \underset{\substack{\text{first}\\\text{digit}}}{1} \cdot \underset{\substack{\text{second}\\\text{digit}}}{2} \cdot \underset{\substack{\text{third}\\\text{digit}}}{9}$$

8. How many arrangements

(a) Fluke all letters different $\to 5!$

(b) Propose 2 Ps, 2 Os $\to \frac{7!}{2!2!}$

(c) Mississippi 4 Is, 4 Ss, 2 Ps $\to \frac{11!}{4!4!2!}$

(d) Arrange 2 As, 2 Rs $\to \frac{7!}{2!2!}$

9. 12 blocks, 6 black, 4 red, 1 white, 1 blue

$\#$ arrangements $= \frac{12!}{6!4!1!1!} = \frac{12!}{6!4!}$

11. Arranging books 3 novels, 2 math, 1 chem

(a) Books in any order $\to 6!$

(b) math books together $\to 3! \cdot 3! \cdot 2! \cdot 1!$
novels together
(chem together since only 1)

$3!$: order of subjects; $3!$: order of novels; $2!$: order of math; $1!$: chem

(c) Novels together, others in any order

Think of novels as one block; we're arranging the other 3 books and this one block

$\rightarrow$ 4! arrangements

The block of novels can be arranged in 3! ways

Total $= 4! \cdot 3!$

13. 20 people: everyone shakes hands with everyone

$\#$ handshakes $= \binom{20}{2}$ since each handshake involves an unordered set of 2 people

28. 8 teachers divided between 4 schools.

There are 4 choices for where each teacher can go.

$$\underbrace{4}_{\text{first teacher}} \cdot \underbrace{4}_{\text{second teacher}} \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^8$$

, etc.

What if each school recieves 2 teachers?
First school get 2 of 8, second gets 2 of the six that remain, so # of way is

$$\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{8!}{6!\,2!}\,\frac{6!}{4!\,2!}\,\frac{4!}{2!\,2!} = \frac{8!}{(2!)^4}$$

Theoretical Exercises

2. Two experments: first has m outcomes
   # of outcomes of second experiment depends on the outcome of first experiment:

   if first experment has outcome i, then second experiment has n_i outcomes

   Table of outcomes of the pair:

   $(1,1)$ $(1,2)$ $\cdots$ $(1,n_1)$

   $(2,1),(2,2)$ $\cdots\cdots$ $(2,n_2)$

   $\vdots$

   $(m,1),(m,2)$ $\cdots$ (m,n_m)

   n_i outcomes in ith row:

   $$\text{total} = n_1+n_2+\cdots+n_m = \sum_{i=1}^{m} n_i$$

3. Select r objects from n; order of selection is relevent:

   $$\underbrace{n}_{\text{first object chosen}} \cdot \underbrace{(n-1)}_{\text{second object chosen}} \cdot \underbrace{(n-2)}_{\text{3rd}} \cdots \underbrace{(n-(r-1))}_{r\text{th}}$$

   when choosing rth object $(r-1)$ are gone

Don't divide by $r!$ because order <u>does</u> matter!

4. There are $\binom{n}{r}$ arrangements of n balls of which r are black and $n-r$ are white

Prove this combinatorially:

When arranging the balls, we have n places to put them in. Once we choose which places contain black balls, the placement of the white balls is determined (the white balls go in the places left over)

so $\#(\text{arrangements}) = \#\left(\begin{matrix}\text{choose } r \text{ places out of } n\\ \text{(where the black balls go)}\end{matrix}\right)$

$= n \text{ choose } r = \binom{n}{r}$

8. Prove $\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}$

<u>Proof</u>: consider a group of $n+m$ people, n men
m women

$\#$ groups of r people $= \binom{n+m}{r}$

On the other hand, we can seperate out the groups that have all women, 1 men and $r-1$ women, 2 men and $r-2$ women, etc.

groups with

0 men, r women → $\binom{n}{0}\binom{m}{r}$

1 men, r-1 women → $\binom{n}{1}\binom{m}{r-1}$

2 men, r-2 women $\binom{n}{2}\binom{m}{r-2}$

⋮

r-1 men, 1 women $\binom{n}{r-1}\binom{m}{1}$

r men, 0 women $\binom{n}{r}\binom{m}{0}$

Adding these up gives all possible $\binom{n+m}{r}$ groups

So: $\binom{n}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \dots + \binom{n}{r}\binom{m}{0}$

10. Group of n people: choose committee of size k, and a chairperson for this committee.

Count possibilities 3 ways

(a) committee first, then chair

$$\left.\begin{array}{l}\#\text{ committees } = \binom{n}{k} \\ \#\text{ possible chairs, given committee} = k\end{array}\right\} \to \binom{n}{k}\cdot k$$

(b) choose committee minus chair, then chair

committees minus chair = # groups of k-1 out of n

$$= \binom{n}{k-1}$$

chairs, given that we've chosen k-1 non-chairs

$$= (n-(k-1)) = (n-k+1)$$

$$\Rightarrow \binom{n}{k-1}(n-k+1)$$

(c) choose chair, then choose rest of committee

$$\#\text{chairs} = n$$

groups of k-1 for rest of committee, given one person is already chosen to be chair

$$= \binom{n-1}{k-1}$$

$$\Rightarrow n\binom{n-1}{k-1}$$

(d) Since we counted the same thing 3 ways, they must be equal

$$\binom{n}{k}k = \binom{n}{k-1}(n-k+1) = n\binom{n-1}{k-1}$$

(e) To verify this using the formula:

$$k\binom{n}{k} = k\frac{n!}{(n-k)!\,k!} = \cancel{k}\,\frac{n!}{(n-k)!\,(\cancel{k}\cdot(k-1)\cdots 2\cdot 1)}$$

$$= \frac{n!}{(n-k)!\,(k-1)!}$$

$$\binom{n}{k-1}(n-k+1) = \frac{n!}{(n-(k-1))!\,(k-1)!}(n-k+1)$$

$$= \frac{n!}{\cancel{(n-k+1)}(n-k)!\,(k-1)!}\cancel{(n-k+1)}$$

$$= \frac{n!}{(n-k)!\,(k-1)!}$$

$$n\binom{n-1}{k-1} = n\frac{(n-1)!}{[(n-1)-(k-1)]!\,(k-1)!} = n\frac{(n-1)!}{(n-k)!(k-1)!}$$

$$= \frac{n!}{(n-k)!\,(k-1)!}$$

So all three equal $\dfrac{n!}{(n-k)!\,(k-1)!}$ ✓

Solutions 2

Ch 1 Theoretical exercises

13. Show, for $n>0$ $\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$

Use Binomial theorem $\sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} = (x+y)^n$

Plug in $x=-1$ and $y=1$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i (1)^{n-i} = (-1+1)^n = 0^n = 0$$

$(1)^{n-i}=1$ so get $\sum_{i=0}^{n} \binom{n}{i} (-1)^i = 0$ QED

14. From set of n people committee of size j and a subcommittee of size i are chosen.

(a) choose committee : $\binom{n}{j}$

then choose subcommittee there of : $\binom{j}{i}$

so get $\binom{n}{j}\binom{j}{i}$

choose smaller subcommittee first $\binom{n}{i}$

then choose rest of committee $\binom{n-i}{j-i}$

so get $\binom{n}{i}\binom{n-i}{j-i}$

Thus $\binom{n}{j}\binom{j}{i} = \binom{n}{i}\binom{n-i}{j-i}$

(b) Prove $\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} = \binom{n}{i} 2^{n-i} \quad (i \leq n)$

Consider $\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} = \sum_{j=i}^{n} \binom{n}{i}\binom{n-i}{j-i}$ by part (a)

$= \binom{n}{i} \sum_{j=i}^{n} \binom{n-i}{j-i}$ since $\binom{n}{i}$ is independent of j

Now $\sum_{j=i}^{n} \binom{n-i}{j-i} = \binom{n-i}{0} + \binom{n-i}{1} + \cdots + \binom{n-i}{n-i}$

so $= \sum_{k=0}^{n-i} \binom{n-i}{k}$ (reindexing)

But $\sum_{k=0}^{n-i} \binom{n-i}{k} = 2^{n-i}$

In general $\sum_{k=0}^{N} \binom{N}{k} = 2^N$ by binomial thm see p. 9, Example 4e

So get $\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} = \binom{n}{i} 2^{n-i}$ QED.

(c) Prove $\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i}(-1)^{n-j} = 0$

$\sum_{j=1}^{n} \binom{n}{j}\binom{j}{i}(-1)^{n-j} = \sum_{j=i}^{n} \binom{n}{i}\binom{n-i}{j-i}(-1)^{n-j}$ by (a)

$= \binom{n}{i} \sum_{j=i}^{n} \binom{n-i}{j-i}(-1)^{n-j}$

So Need To Show $\sum_{j=i}^{n} \binom{n-i}{j-i} (-1)^{n-j} = 0$

this is true by exercise 13

set $N = n-i$, $K = j-i$, then $n-j = N-K$

so $\sum_{j=i}^{n} \binom{n-i}{j-i} (-1)^{n-j} = \sum_{K=0}^{N} \binom{N}{K} (-1)^{N-K}$

$= \sum_{K=0}^{N} \binom{N}{K} (-1)^{N-K} (1)^{K} = (1-1)^{N} = 0.$ QED.

Ch 2 problems

1. Box with 1 red, 1 green, 1 blue marble

Experiment I: draw marble, replace, draw again

S = all pairs of colors = { (R,R) (R,G) (R,B)
(G,R) (G,G) (G,B)
(B,R) (B,G) (B,B) }

$3^2 = 9$ points in S.

Experiment II: draw marble, don't replace, draw again

S = all pairs of distinct colors = { (R,G) (R,B)
(G,R) (G,B)
(B,R) (B,G) }

$3 \cdot 2 = 6$ points in S.

2. Die is rolled until a 6 appears, then stop

$S = \{$ any sequence of digits chosen from 1–5, and end on a 6 $\}$

e.g. 121351426 or 115436
or 1111 ⋯ (just 1's forever)
or 123123123⋯ forever.

E_n = event that experiment lasts n rolls
= $\{$ sequences of length n $\}$

e.g. 115436 is in E_6

$\bigcup_{n=1}^{\infty} E_n$ = event that experiment eventually ends

$\left(\bigcup_{n=1}^{\infty} E_n\right)^c$ = event that experiment never ends
= event that we never get a 6.

3. Two dice thrown
E = sum of dice is odd
F = at least one die is 1
G = sum of dice is 5.

EF = sum is odd and at least one die is 1
$E \cup F$ = sum is odd or one die is 1 or both
FG = sum of dice is 5 and at least on is 1
= one die is 1 and the other is 4.
EF^c = sum is odd and neither die is 1.
EFG = sum is 5 and one die is 1 = FG (since 5 implies odd
$G \subset E$)

4. A, B, C take turns fliping a coin (A first, B second, C third) first to get heads wins.

Sample space is $S = \begin{cases} 1, 01, 001, 0001, \ldots \\ 000\cdots \end{cases}$

(a) Interpret sample space:

0 represents tails, 1 heads, sequece is A's flip, B's flip, C's flip, A's flip, etc.

(b) Define in terms of S:

(i) A wins $= \{1, 0001, 0000001, \ldots\}$

$= \{3k$ 0's, followed by 1, for any $k = 0, 1, 2, \ldots\}$

(ii) B wins $= \{01, 00001, 00000001, \ldots\}$

$= \{3k+1$ 0's, followed by 1, for any $k = 0, 1, \ldots\}$

(iii) $(A \cup B)^c = A^c B^c$

$= \{001, 000001, \ldots\} \cup \{000\cdots\}$

$= \{3k+2$ 0's, followed by 1$\} \cup \{$infinitely many 0's$\}$

= C wins or game never ends.

5. System has 5 components

Let $(x_1, x_2, x_3, x_4, x_5)$ be a vector denoting whether each component is working or not

0 → not working
1 → working

So $(1,1,0,0,1)$ means 1,2,5 working
3,4 not

(a) How many outcomes? each x_i either 0 or 1 → $2^5 = 32$

(b) W = (1 and 2 work) or (3 and 4 work) or (1,3,5 work)

Outcomes in W:

1 & 2 working	3 & 4	1 & 3 & 5
1 1 1 1 1	1 1 1 1 1	1 1 1 1 1
1 1 1 1 0	1 1 1 1 0	1 1 1 0 1
1 1 1 0 1	1 0 1 1 1	1 0 1 1 1
1 1 0 1 1	0 1 1 1 1	1 0 1 0 1
1 1 1 0 0	1 0 1 1 0	
1 1 0 1 0	0 1 1 1 0	
1 1 0 0 1	0 0 1 1 1	
1 1 0 0 0	0 0 1 1 0	

= Duplicate

15 outcomes

(c) A = 4 and 5 not working
outcomes in A = $2^3 = 8$ since other 3 parts can be in any state.

(d) Outcomes in AW: look at list for (b), see which end in 00:

11100 — 2 outcomes in AW.
11000

Ch 2 Theoretical exercises.

Prove the relation:

1. $EF \subset E \subset E \cup F$

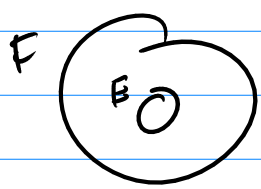
if E and F occur, then E occurs so $EF \subset E$

If E occurs, then E or F or both occur so $E \subset E \cup F$

(could also use Venn diagram)

2. If $E \subset F$, then $F^c \subset E^c$

If E occurs, then F occurs. So if F does not occur, then E cannot occur, so $F^c \subset E^c$.

Venn: 

outside F implies outside E

3. $F = FE \cup FE^c$ and $E \cup F = E \cup E^c F$

Proof: $F = FS = F(E \cup E^c) = FE \cup FE^c$ (distributive law)

QED

For second part, use $F = FE \cup FE^c$ and take union with E

$$E \cup F = E \cup (FE \cup FE^c) = (E \cup FE) \cup FE^c$$

Now, $E \cup FE = E$, since $FE \subset E$ (by exercise 1).

so $E \cup F = E \cup FE^c$ QED.

6. E, F, G three events

(a) only $E = EF^c G^c$

(b) both E & G, not F $= EGF^c$

(c) at least one $= E \cup F \cup G$

(d) at least two $= EF \cup EG \cup FG$

(e) all three $= EFG$

(f) none $= E^c F^c G^c = (E \cup F \cup G)^c$

(g) at most one $=$ (at least two)$^c = (EF \cup EG \cup FG)^c$
$=$ at least two of E^c, F^c, G^c $= E^cF^c \cup E^cG^c \cup F^cG^c$

(h) at most two $=$ (all three)$^c = (EFG)^c$
$= E^c \cup F^c \cup G^c =$ at least one of E^c, F^c, G^c.

(i) exactly two $= EFG^c \cup EF^cG \cup E^cFG$

(j) At most 3 $= S$

7. Simplify:

(a) $(E\cup F)(E\cup F^c) = EE \cup EF \cup EF^c \cup FF^c$ (distributive law)

$= E \cup EF \cup EF^c \cup \emptyset = \boxed{E}$ since EF and EF^c are contained in E.

(b) $(E\cup F)(E^c\cup F)(E\cup F^c)$

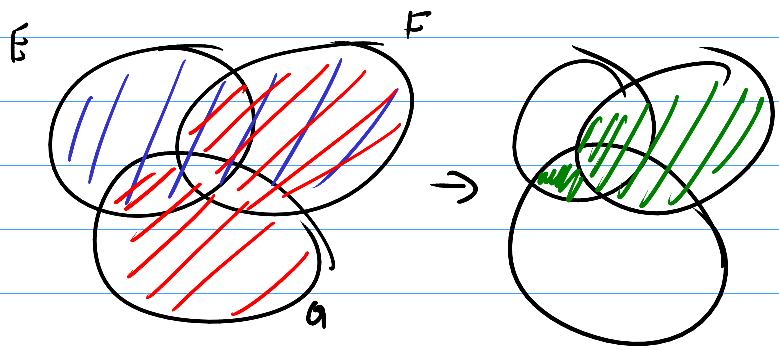
$= \underbrace{(E\cup F)(E\cup F^c)}(E^c\cup F)$ commutative law

$= E \quad (E^c\cup F)$ by (a)

$= EE^c \cup EF = \emptyset \cup EF = EF$

(c) $(E\cup F)(F\cup G) = EF\cup FF\cup EG\cup FG$ distributive

$= \underbrace{EF\cup FG}_{\text{these are contained in } F} \cup F \cup EG = F\cup EG$



9. Run experiment n times, get outcomes $x_1, \dots, x_n$

for event E, define $n(E)$ number of times E occured in these trials = # of x_i such that x_i is in E

Define $f(E) = \frac{n(E)}{n}$. We show it satisfies Axioms 1,2,3:

Axiom 1: By definition $0 \le n(E) \le n$ so

$$0 \le \frac{n(E)}{n} \le 1$$

Axiom 2: Since S occured n times, $n(S) = n$

so $f(S) = 1$

Axiom 3: E_1, E_2, E_3, mutually exclusive $E_i E_j = \emptyset$

$n\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} n(E_i)$ since each time E_i occurs in the sequence $x_1, \dots, x_n$, the other E_j do not occur.

So $f\left(\bigcup_{i=1}^{\infty} E_i\right) = \frac{n\left(\bigcup_{i=1}^{\infty} E_i\right)}{n} = \sum_{i=1}^{\infty} \frac{n(E_i)}{n} = \sum_{i=1}^{\infty} f(E_i)$ $\blacksquare$

Solutions 3
Ch 2 Problems

8. A, B mutually exclusive $P(A) = .3$ $P(B) = .5$

(a) $P(A \cup B) = P(A) + P(B) - P(AB) = .3 + .5 - 0 = .8$

Since $P(AB) = P(\emptyset) = 0$

(b) A but not $B = AB^c$

Now $A = AS = A(B \cup B^c) = AB \cup AB^c = \emptyset \cup AB^c = AB^c$

$A = AB^c$

$P(AB^c) = P(A) = .3$

(c) both A and B: $P(AB) = P(\emptyset) = 0$

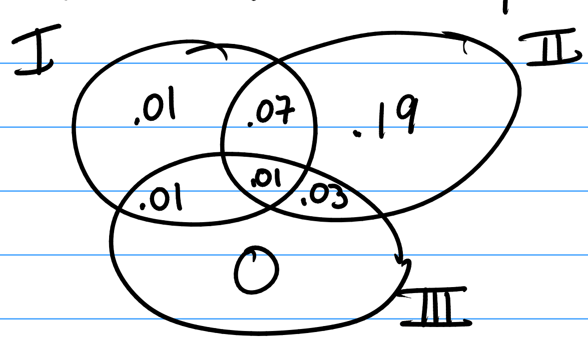
13. Let E_I, E_{II} and E_{III} denote events that a random person reads each of the newspapers I, II, III

So we know $P(E_I) = .1$ $P(E_{II}) = .3$ $P(E_{III}) = .05$

$P(E_I E_{II}) = .08$ $P(E_I E_{III}) = .02$ $P(E_{II} E_{III}) = .04$

$P(E_I E_{II} E_{III}) = .01$

We can solve for all parts of the venn diagram



e.g. $P(E_I E_{II} E_{III}^c) = P(E_I E_{II}) - P(E_I E_{II} E_{III})$

$= .07$

(a) $P(\text{read } \underline{\text{only}} \text{ one}) = .01 + .19 + 0 = .20$

$.20 \times 100000 = 20000$ people read only one

(b) $P(\text{read at least } 2) = .07 + .01 + .01 + .03 = .12$

$.12 \times 100000 = 12000$ people read at least 2

(c) $P(\text{I or III and II}) = P((E_{I} \cup E_{III}) E_{II})$

$= P(E_{I}E_{II} \cup E_{II}E_{III}) = .07 + .03 + .01 = .11$

$.11 \times 100000 = 11000$ people read (I or III) and II

(d) $P((E_{I} \cup E_{II} \cup E_{III})^c) = 1 - P(E_{I} \cup E_{II} \cup E_{III})$

$= 1 - [.01 + .07 + .19 + .01 + .01 + .03] = 1 - .32 = .68$

$.68 \times 100000 = 68000$ people do not read.

(e) $P((\text{I or III but not both}) \text{ and II}) = .07 + .03 = .10$

$.10 \times 100000 = 10000$ people

18. Two cards selected from 52-card deck
Blackjack = A and (10, J, Q, or K)

There are 4 aces in the deck and 16 (10, J, Q or K)'s
so there are $4 \cdot 16$ Blackjack hands

If we assume all $\binom{52}{2}$ hands are equally likely, we get

$$P(\text{Blackjack}) = \frac{4 \cdot 16}{\binom{52}{2}} \approx 0.048$$

22. Shuffle n cards: flip coin n times: if H leave card and move on, if T, move current card to back of deck and move on.

$S = \{$ all H-T sequences of length $n \}$ # outcomes in $S = 2^n$

Outcomes which leave deck unchanged:

$$\left.\begin{array}{l} HH \cdots HHH \\ HH \cdots HHT \\ HH \cdots HTT \\ \vdots \\ HT \cdots TTT \\ TT \cdots TTT \end{array}\right\}$$

in general: get heads for a while, deck doesn't change up to that point. After first tails, must get all tails from then on so that all cards cycle through back to original position.

Such a sequence consists of k heads in a row, followed by $n-k$ tails in a row $k=0,\dots,n$

So there are $n+1$ such sequences

$$P(\text{deck ends up in same order}) = \frac{n+1}{2^n}$$

25. A pair of dice is rolled until a sum of 5 or 7
Find probability that 5 occurs first.

Consider just one trial, where two dice are thrown
36 total outcomes.
4 outcomes result in a sum of 5
6 outcomes result in a sum of 7
26 outcomes result in neither 5 nor 7.

E_n = no 5 or 7 on first $n-1$ rolls, 5 on nth roll.

$$P(E_n) = \underbrace{\frac{26}{36}\cdot\frac{26}{36}\cdot\ \cdots\ \cdot\frac{26}{36}}_{n-1}\cdot\frac{4}{36} = \left(\frac{26}{36}\right)^{n-1}\frac{4}{36}$$

Let E = 5 occurs before 7
If an outcome x is in E, the 5 must occur on the nth roll (for some value of n), so x is in E_n for some n.

$$E = \bigcup_{n=1}^{\infty} E_n$$ Now the events E_n are clearly mutually exclusive, so

$$P(E) = P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty}\left(\frac{26}{36}\right)^{n-1}\frac{4}{36}$$

$$\sum_{n=1}^{\infty}\left(\frac{26}{36}\right)^{n-1}\frac{4}{36} = \frac{4}{36}\,\frac{1}{1-\left(\frac{26}{36}\right)} = \frac{4}{10}$$

[recall geometric series: $\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r}$ if $|r|<1$]

32. b boys, g girls lined up in row
Each of $(b+g)!$ orderings is equally likely

Number girls by $j=1,2,\cdots,g$

Let E_j = event that person in ith spot is girl j

Then E_j consists of $(b+g-1)!$ outcomes (ordering of rest)

So $P(E_j) = \frac{(b+g-1)!}{(b+g)!} = \frac{1}{b+g}$

E = event that person in ith spot is a girl

$E = E_1 \cup E_2 \cup \cdots \cup E_g$.

The events E_j are mutually exclusive, so

$$P(E) = P(E_1 \cup \cdots \cup E_g) = \sum_{j=1}^{g} P(E_j) = \sum_{j=1}^{g} \frac{1}{b+g} = \frac{g}{b+g}$$

39. 5 hotels, 3 people check in.
P(each checks into different hotel) = ?

<u>Assume</u> that each person will choose between the 5 hotels independently, and is equally likely to go to any of the 5.

There are 5^3 outcomes, each with probability $\frac{1}{5^3}$
There are $5 \cdot 4 \cdot 3$ outcomes where no two choose same hotel,

so $P(\text{no two check into same hotel}) = \frac{5 \cdot 4 \cdot 3}{5^3}$

41. Die is rolled 4 times. Probability that 6 comes up at least once

Let E_i denote event that the ith roll is a 6. $(i=1,2,3,4)$

We want $P(E_1 \cup E_2 \cup E_3 \cup E_4)$

$= P(E_1) + P(E_2) + P(E_3) + P(E_4)$

$- P(E_1E_2) - P(E_1E_3) - P(E_1E_4) - P(E_2E_3) - P(E_2E_4) - P(E_1E_4)$

$+ P(E_1E_2E_3) + P(E_1E_2E_4) + P(E_1E_3E_4) + P(E_2E_3E_4)$

$- P(E_1E_2E_3E_4)$

$P(E_i) = \frac{1}{6}$ 4 terms like this

$P(E_iE_j) = \frac{1}{6^2}$ $6 = \binom{4}{2}$ terms like this

$P(E_iE_jE_k) = \frac{1}{6^3}$ $4 = \binom{4}{3}$ terms like this

$P(E_1E_2E_3E_4) = \frac{1}{6^4}$ $1 = \binom{4}{1}$ term like this

Total $= 4 \cdot \frac{1}{6} - 6 \cdot \frac{1}{6^2} + 4\frac{1}{6^3} - \frac{1}{6^4} = P(E_1 \cup E_2 \cup E_3 \cup E_4)$

54. Probability that a bridge hand is void in at least one suit.

There are $\binom{52}{13}$ bridge hands, each equally likely.

V_1 = event that hand is void in spades
V_2 = hand is void in hearts
V_3 = hand is void in diamonds
V_4 = hand is void in clubs

We want $P(V_1 \cup V_2 \cup V_3 \cup V_4)$
use inclusion-exclusion

$$P(V_i) = \frac{\binom{39}{13}}{\binom{52}{13}}$$ since we must form hand out of 39 cards not of a particular suit.

$$P(V_iV_j) = \frac{\binom{26}{13}}{\binom{52}{13}}$$ can only use 26 cards not of suits i or j

$$P(V_iV_jV_k) = \frac{\binom{13}{13}}{\binom{52}{13}}$$ there is only one hand void in suits i, j, k; it consists of the fourth suit

$P(V_1V_2V_3V_4) = 0$ No hand is void in all 4 suits

$$\text{so } P(V_1 \cup V_2 \cup V_3 \cup V_4) = \binom{4}{1}\frac{\binom{39}{13}}{\binom{52}{13}} - \binom{4}{2}\frac{\binom{26}{13}}{\binom{52}{13}} + \binom{4}{3}\frac{\binom{13}{13}}{\binom{52}{13}} - 0$$

$$= \left[4\binom{39}{13} - 6\binom{26}{13} + 4\right] \Big/ \binom{52}{13}$$

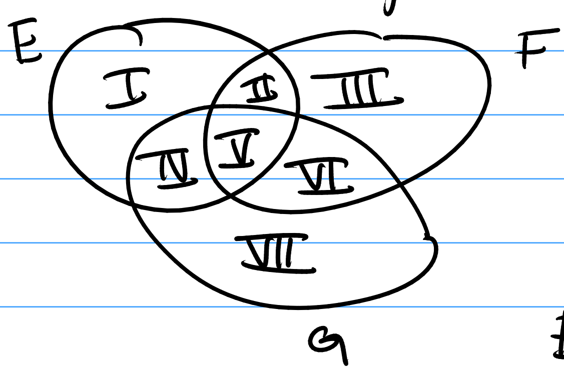
Ch 2 theoretical exercises.

10. Prove $P(E \cup F \cup G) = P(E) + P(F) + P(G)$

$$- P(E^cFG) - P(EF^cG) - P(EFG^c)$$

$$- 2P(EFG)$$

Break Venn diagram into 7 regions



$E = \mathrm{I} \cup \mathrm{II} \cup \mathrm{IV} \cup \mathrm{V}$

$F = \mathrm{II} \cup \mathrm{III} \cup \mathrm{V} \cup \mathrm{VI}$

$G = \mathrm{IV} \cup \mathrm{V} \cup \mathrm{VI} \cup \mathrm{VII}$

$E^cFG = \mathrm{VI}$

$EF^cG = \mathrm{IV}$ $\qquad EFG = \mathrm{V}$

$EFG^c = \mathrm{II}$

so $P(E) + P(F) + P(G) = P(\mathrm{I}) + \cancel{2}P(\mathrm{II}) + P(\mathrm{III}) + \cancel{2}P(\mathrm{IV}) + \cancel{3}P(\mathrm{V}) + \cancel{2}P(\mathrm{VI}) + P(\mathrm{VII})$

$- P(E^cFG) - P(EF^cG) - P(EFG^c) = - \cancel{P(\mathrm{VI})} - \cancel{P(\mathrm{IV})} - \cancel{P(\mathrm{II})}$

$- 2P(EFG) \qquad = - \cancel{2P(\mathrm{V})}$

Add up $\qquad P(\mathrm{I}) + P(\mathrm{II}) + P(\mathrm{III}) + P(\mathrm{IV}) + P(\mathrm{V}) + P(\mathrm{VI}) + P(\mathrm{VII})$

the expression at the bottom equals $P(E \cup F \cup G)$
so we are done.

11. If $P(E) = .9$ and $P(F) = .8$ show $P(EF) \geq .7$
in general show
$$P(EF) \geq P(E) + P(F) - 1$$

Proof By inclusion-exclusion
$$P(E \cup F) = P(E) + P(F) - P(EF)$$

by axiom 1, $P(E \cup F) \leq 1$

so $1 \geq P(E) + P(F) - P(EF)$

$$P(EF) \geq P(E) + P(F) - 1$$

In the special case $P(EF) \geq .9 + .8 - 1 = .7$

18. f_n = # of way of tossing a coin n times so that
no successive heads appear.

Clearly $f_1 = 2$ $f_2 = 3$

Consider f_n $n > 3$
if we get a tails first, there are f_{n-1} ways of
doing the rest $(n-1)$ flips so that no successive heads
appear
if we get a heads first, we need to get a tails
second, and then there are f_{n-2} ways of
doing the rest $(n-2)$ flips so that no successive

heads appear. Thus $f_n = f_{n-1} + f_{n-2}$

let P_n denote the probability of no successive heads when a coin is flipped n times.

$P_n = \dfrac{f_n}{2^n}$ since all 2^n outcomes are equally likely

What is P_{10}

$$\left.\begin{aligned}
&f_1 = 2\\
&f_2 = 3\\
&f_3 = 2+3=5\\
&f_4 = 3+5=8\\
&f_5 = 5+8=13\\
&f_6 = 8+13=21\\
&f_7 = 13+21=34\\
&f_8 = 21+34=55\\
&f_9 = 34+55=89
\end{aligned}\right\}\ \text{Cf. Fibonacci numbers}$$

$f_{10} = 55+89 = 144$ $\qquad P_{10} = 144/2^{10} = 144/1024$

19. An Urn contains n red and m blue balls withdrawn one at a time until r $(r \leq n)$ red balls are gotten. find probability that k balls are withdrawn.

k balls are withdrawn if $r-1$ red balls are among the first $k-1$, and the kth ball is red

$\boxed{\text{NOTE: This solution is using conditional probability}}$

Prob. to choose $r-1$ red out of $k-1$ trys

$$= \frac{\binom{n}{r-1}\binom{m}{(k-1)-(r-1)}}{\binom{n+m}{k-1}} = \frac{\binom{n}{r-1}\binom{m}{k-r}}{\binom{n+m}{k-1}}$$

Prob. to choose kth ball to be red, once $r-1$ red, and $k-1$ total, balls have been removed:

$$= \frac{n-(r-1)}{n+m-(k-1)} = \frac{n-r+1}{n+m-k+1}$$

so over-all: $P(k \text{ balls withdrawn}) = \frac{\binom{n}{r-1}\binom{m}{k-r}}{\binom{n+m}{k-1}} \cdot \frac{n-r+1}{n+m-k+1}$

Ch 3 theoretical exercises

3.2 Let $A \subset B$. Express as simply as possible

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)} \quad \left(AB = A \text{ since } A \subset B\right)$$

$$P(A|B^c) = \frac{P(AB^c)}{P(B^c)} = 0 \quad \left(\begin{array}{c} A \subset B \Rightarrow AB^c \subset BB^c = \emptyset \\ AB^c = \emptyset \end{array}\right)$$

$$P(B|A) = \frac{P(BA)}{P(A)} = \frac{P(A)}{P(A)} = 1 \quad \left(BA = A \text{ since } A \subset B\right)$$

$$P(B \mid A^c) = \frac{P(BA^c)}{P(A^c)}$$

$$= \frac{P(B) - P(A)}{1 - P(A)}$$

$B = BA^c \cup BA = BA^c \cup A$

so $P(B) = P(BA^c) + P(A)$

Hence $P(BA^c) = P(B) - P(A)$

Also $P(A^c) = 1 - P(A)$

Solutions 4

Ch 3 Problems

3.12 Let E_1, E_2, E_3 denote events that student passes 1st, 2nd, and 3rd exams

So $P(E_1) = .9 \quad P(E_2|E_1) = .8 \quad P(E_3|E_1E_2) = .7$

Passes 1st & 2nd: $P(E_2E_1) = P(E_2|E_1)\,P(E_1) = (.8)(.9) = .72$

(a) Pass All $P(E_3E_2E_1) = P(E_3|E_1E_2)\,P(E_1E_2) = (.7)(.72) = .504$

(b) P(failed second | did not pass all)

$$= P\left(E_1E_2^c \,\middle|\, (E_1E_2E_3)^c\right) = \frac{P\left((E_1E_2E_3)^c \,\middle|\, E_1E_2^c\right) P(E_1E_2^c)}{P((E_1E_2E_3)^c)}$$

$$P\left((E_1E_2E_3)^c \,\middle|\, E_1E_2^c\right) = 1$$

(if she fails second exam, she cannot pass all 3)

$$P((E_1E_2E_3)^c) = 1 - P(E_1E_2E_3) = 1 - .504 = .496$$

$$P(E_1E_2^c) = P(E_1) - P(E_1E_2)$$ since $E_1 = E_1E_2 \cup E_1E_2^c$ mutually exclusive

$$= .9 - .72 = .18$$

so Answer $= \dfrac{(1)(.18)}{.496} = .362904$

3.22 Roll red, blue, yellow dice
If R, B, Y are numbers appearing on the dice, we ask for the probability of the event that
$B < Y < R$.

(a) $P($no two dice on same #$)$

Sample space has 6^3 equally likely outcomes
There are $6 \cdot 5 \cdot 4$ outcomes in which no number is repeated

$$\underset{B}{6} \cdot \underset{Y}{5} \cdot \underset{R}{4}$$

so $P(B \neq Y \text{ and } Y \neq R \text{ and } B \neq R) = \dfrac{6 \cdot 5 \cdot 4}{6^3} = \dfrac{10}{18}$

(b) Given no two dice land on same #, what is probability that $B < Y < R$

The reduced sample space consists of $6 \cdot 5 \cdot 4$ outcomes, all equally likely. There are $\binom{6}{3} = \dfrac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1}$ possible sets of numbers that can appear, and each set of numbers can be distributed across the Blue, Yellow, Red dice in $3 \cdot 2 \cdot 1 = 6$ ways. For a given set of 3 numbers there is only one of these 6 ways that satisfies $B < Y < R$, so

$$P(B<Y<R \mid \text{no two same}) = \frac{\left(\frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1}\right)}{6 \cdot 5 \cdot 4} = \frac{1}{3 \cdot 2 \cdot 1} = \frac{1}{6}.$$

(c) $P(B<Y<R) = P(B<Y<R \mid \text{no two same}) P(\text{no two same}) = \dfrac{10}{18} \cdot \dfrac{1}{6} = \dfrac{10}{108}$

3.24 2 balls are either gold or black w/ $\frac{1}{2}$ probability, and independently from each other, placed in urn

4 outcomes each with $\frac{1}{4}$ probability: $\begin{cases} GG \\ GB \\ BG \\ BB \end{cases}$

(a) We know at least one ball is Gold $F = \{GG, GB, BG\}$
Both gold $= \{GG\}$
$P(\text{Both gold} \mid \text{at least one gold}) = P(\{GG\} \mid \{GG, GB, BG\})$

$$= \frac{P(\{GG\})}{P(\{GG, GB, BG\})} = \frac{1/4}{3/4} = \frac{1}{3}$$

(b) A gold ball falls out of urn, so the probability of the other being gold is $\frac{1}{2}$.

$$P(\text{both gold} \mid \text{discovered ball is gold}) = \frac{1}{2}$$

This is an apparent paradox, since it seems in both cases that we know one of the balls is gold. But in the second case, we also have to take into account the chance that the other ball could have fallen out.
So in (b) there are actually 8 possible outcomes

GG	GG	O = which one falls out.
GB	GB	
BG	BG	
BB	BB	

So we see that exactly half the cases where a gold ball falls out, the other ball is gold.

3.37 Gambler has fair coin $\left(P(H) = \frac{1}{2} = P(T)\right)$
and a 2-head coin $\left(P(H) = 1, P(T) = 0\right)$

(a) one coin is selected at random, flipped, comes up heads
what is probability the coin is fair.

$$P(\text{fair} \mid H) = \frac{P(H \mid \text{fair})\, P(\text{fair})}{P(H \mid \text{fair})\, P(\text{fair}) + P(H \mid \text{2-headed})\, P(\text{2-headed})}$$

$P(\text{fair}) = P(\text{2-headed}) = \frac{1}{2}$ since coin selected randomly

$$P(\text{fair} \mid H) = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right)} = \frac{1/4}{1/4 + 1/2} = \frac{1/4}{3/4} = \frac{1}{3}$$

(b) coin flipped again, heads again, what prob. coin is fair.

$$P(\text{fair} \mid HH) = \frac{P(HH \mid \text{fair})\, P(\text{fair})}{P(HH \mid \text{fair})\, P(\text{fair}) + P(HH \mid \text{2-headed})\, P(\text{2-headed})}$$

$$= \frac{\left(\frac{1}{4}\right)\left(\frac{1}{2}\right)}{\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{2}\right)} = \frac{1/8}{1/8 + 1/2} = \frac{1/8}{5/8} = \frac{1}{5}$$

(c) Coin fliped, comes up tails what is probability coin is fair
Obviously the coin must be fair:

$$P(\text{fair} \mid HHT) = \frac{P(HHT \mid \text{fair})\, P(\text{fair})}{P(HHT \mid \text{fair})\, P(\text{fair}) + P(HHT \mid \text{2-head})\, P(\text{2-head})}$$

Since $P(HHT \mid \text{2-headed}) = 0$, get

$P(\text{fair} \mid HHT) = 1$, as expected.

3.47 Urn w/ 5 white, 10 black.
Fair die is rolled, that number of balls is drawn

P(all white?)
Let R_i $(i=1,2,3,4,5,6)$ denote the event that the die rolls the number i
W = event that all balls drawn are white

$$P(W) = \sum_{i=1}^{6} P(W \mid R_i)\, P(R_i), \quad P(R_i) = \frac{1}{6}$$

$$P(W \mid R_i) = \text{prob. of getting } i \text{ white when drawing } i = \frac{\binom{5}{i}}{\binom{15}{i}}$$

$$P(W) = \frac{\binom{5}{1}}{\binom{15}{1}}\frac{1}{6} + \frac{\binom{5}{2}}{\binom{15}{2}}\frac{1}{6} + \frac{\binom{5}{3}}{\binom{15}{3}}\frac{1}{6} + \frac{\binom{5}{4}}{\binom{15}{4}}\frac{1}{6} + \frac{\binom{5}{5}}{\binom{15}{5}}\frac{1}{6} + \frac{\binom{5}{6}}{\binom{15}{6}}\frac{1}{6}$$

$\binom{5}{6} = 0$

$$= \frac{1}{6}\left[\frac{5}{15} + \frac{5\cdot4}{15\cdot14} + \frac{5\cdot4\cdot3}{15\cdot14\cdot13} + \frac{5\cdot4\cdot3\cdot2}{15\cdot14\cdot13\cdot12} + \frac{5\cdot4\cdot3\cdot2\cdot1}{15\cdot14\cdot13\cdot12\cdot11} + 0\right]$$

$$= \frac{1}{6}\cdot\frac{5}{11}$$

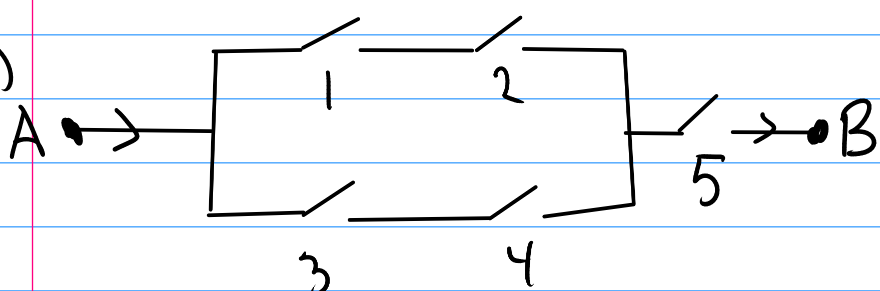
$$P(R_3 \mid W) = \frac{P(W \mid R_3)\,P(R_3)}{P(W)}$$

$$= \left[\frac{\binom{5}{3}}{\binom{15}{3}}\right] \cdot \frac{1}{6} \Big/ P(W) = \left(\frac{2}{91}\right)\cdot\frac{1}{6} \Big/ \left(\frac{5}{11}\right)\cdot\frac{1}{6}$$

$$= \frac{22}{455}$$

3.66 Let E_i denote the event that ith relay closes. These events are independent and $P(E_i) = p_i$

(a)



Current flows $\Leftrightarrow$ (1 & 2 close or 3 & 4 close) and 5 closes

$$= (E_1E_2 \cup E_3E_4)E_5$$

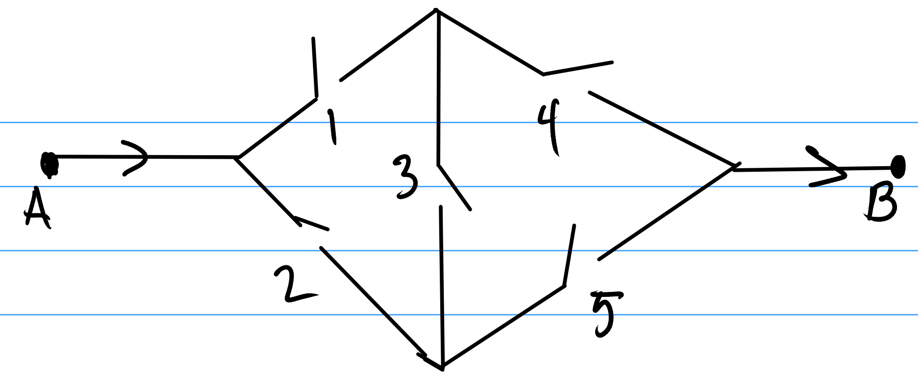
$$P((E_1E_2 \cup E_3E_4)E_5) = P(E_1E_2 \cup E_3E_4)\,P(E_5) \text{ by independence}$$

$$P(E_1E_2 \cup E_3E_4) \underset{\substack{\uparrow \\ \text{by inclusion exclusion}}}{=} P(E_1E_2) + P(E_3E_4) - P(E_1E_2E_3E_4)$$

$$= p_1p_2 + p_3p_4 - p_1p_2p_3p_4 \quad \text{by independence}$$

$$P(\text{current flows}) = [p_1p_2 + p_3p_4 - p_1p_2p_3p_4]\,p_5$$

(b)



Condition on E_3:

If 3 closes, need (1 or 2) and (4 or 5)

$$P(\text{current flows} \mid E_3) = P((E_1 \cup E_2)(E_4 \cup E_5))$$

$$= P(E_1 \cup E_2)\, P(E_4 \cup E_5) \quad \text{by independence}$$

$$= [P(E_1) + P(E_2) - P(E_1E_2)][P(E_4) + P(E_5) - P(E_4E_5)]$$

$$= [p_1 + p_2 - p_1p_2][p_4 + p_5 - p_4p_5]$$

If 3 is open, need (1 and 4) or (2 and 5)

$$P(\text{current flows} \mid E_3^c) = P(E_1E_4 \cup E_2E_5)$$

$$= P(E_1E_4) + P(E_2E_5) - P(E_1E_4E_2E_5)$$

$$= p_1p_4 + p_2p_5 - p_1p_2p_4p_5$$

$$P(\text{current flows}) = P(\text{current flows} \mid E_3) P(E_3) + P(\text{current flows} \mid E_3^c) P(E_3^c)$$

$$= [p_1 + p_2 - p_1p_2][p_4 + p_5 - p_4p_5]p_3 + [p_1p_4 + p_2p_5 - p_1p_2p_4p_5](1 - p_3)$$

3.78 A and B play series of games
Each game is independent, $P(A \text{ wins}) = p$
$P(B \text{ wins}) = 1-p$

Match ends when one player has won 2 more games than the other has won.

Note that a match always has an even # of games: A wins n times, B wins n+2 times, or vice versa, so 2n+2 games are played

(a) P(4 games are played)

Length of match:	2	4
possible sequence of wins	AA	ABAA
	BB	ABBB
		BAAA
		BABB

Here, "A" denotes a win for A, B same.

$$P(\text{Match lasts 4 games}) = P(ABAA) + P(ABBB) + P(BAAA) + P(BABB)$$
$$= p^3(1-p) + p(1-p)^3 + p^3(1-p) + p(1-p)^3$$
$$= 2\left[p^3(1-p) + p(1-p)^3\right]$$

(b) P(A wins match)

The rules of this game are similar to the rules for a tie game of Tennis ("deuce"/"advantage" system)

We group the games into consecutive pairs:
Starting from a tie 4 things can happen

AA   Tie $\xrightarrow{A}$ advantage A $\xrightarrow{A}$ A wins match

AB   Tie $\xrightarrow{A}$ advantage A $\xrightarrow{B}$ tie

BA   Tie $\xrightarrow{B}$ advantage B $\xrightarrow{A}$ tie

BB   Tie $\xrightarrow{B}$ advantage B $\xrightarrow{B}$ B wins match.

So starting from tie, each sequence of two games results in either a tie or a win for one side.

So A wins $\Leftrightarrow$ some sequence of BA or AB pairs, followed by AA

Let E_i = AB or BA on games $2i-1$ and $2i$

F_i = AA on games $2i-1$ and $2i$

$$P(E_i) = \underset{A\ B}{p(1-p)} + \underset{B\ A}{(1-p)p} = 2p(1-p)$$

$$P(F_i) = p^2$$

These events are independent for different values of i.

A wins in $2n+2$ games $= E_1 E_2 \cdots E_n F_{n+1}$

$$P(\text{A wins in } 2n+2 \text{ games}) = [2p(1-p)]^n p^2$$

$\{A \text{ wins}\} = \bigcup_{n=0}^{\infty} \{A \text{ wins in } 2n+2 \text{ games}\}$

$$P(A \text{ wins}) = \sum_{n=0}^{\infty} [2p(1-p)]^n p^2$$

$$= p^2 \sum_{n=0}^{\infty} [2p(1-p)]^n = p^2 \frac{1}{1-(2p(1-p))} = \frac{p^2}{1-2p+2p^2}$$

$$= \frac{p^2}{p^2+(1-p)^2}.$$

3.4 Ball in one of n boxes $P(\text{in } i\text{th box}) = P_i$
search of ith box finds it with probability α_i

Let E_i = ball is in ith box
S_i = search of ith box finds ball.

$$P(E_j \mid S_i^c) = \frac{P(S_i^c \mid E_j)\, P(E_j)}{\sum_{k=1}^{n} P(S_i^c \mid E_k) P(E_k)}$$

$P(E_j) = P_j$ $\qquad P(S_i^c \mid E_j) = \begin{cases} 1 & \text{if } i \neq j \\ 1-\alpha_i & \text{if } i = j \end{cases}$

Thus $\displaystyle\sum_{k=1}^{n} P(S_i^c \mid E_k) P(E_k) = \sum_{k \neq i} P_k + (1-\alpha_i)P_i$

$$= \left(\sum_{k=1}^{n} P_k\right) - \alpha_i P_i = 1 - \alpha_i P_i$$

So $P(E_j | S_i^c) = \dfrac{P(S_i^c | E_j) P_j}{1 - \alpha_i P_i} = \begin{cases} \dfrac{P_j}{1-\alpha_i P_i} & j \neq i \\ \dfrac{(1-\alpha_i)P_i}{1-\alpha_i P_i} & j = i \end{cases}$ $\boxed{//}$

3.6 Suppose $E_1, E_2, E_3, \ldots, E_n$ are independent

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = P([E_1^c E_2^c \cdots E_n^c]^c) \text{ by DeMorgan}$$

$$= 1 - P(E_1^c E_2^c \cdots E_n^c)$$

$$= 1 - \prod_{i=1}^{n} P(E_i^c) \quad \text{by independence}$$

$$= 1 - \prod_{i=1}^{n} [1 - P(E_i)]$$

3.10 n people $A_{i,j}$ = persons i and j have same birthday

$$P(A_{i,j}) = \frac{1}{365}$$

The events $A_{i,j}$ & $A_{r,s}$ for different pairs are independent

if $\{i,j\} \cap \{r,s\} = \emptyset$ this is obvious

otherwise, we are looking at $A_{i,j}$ and $A_{j,k}$

$P(A_{j,k} | A_{i,j})$ = Prob that k has birthday on the particular day which i and j share =

$= \frac{1}{365} = P(A_{j,k})$

But what about the three events $A_{i,j}, A_{j,k}, A_{i,k}$

If these are to be independent, we need

$P(A_{i,k} \mid A_{i,j} A_{j,k}) = \frac{1}{365}$, But No!

i and j share birthday, and j and k share birthday, then
i and k share birthday, necessarily

so in fact $P(A_{i,k} \mid A_{i,j} A_{j,k}) = 1 \neq \frac{1}{365}$

And these events are not independent all together.

3.16 Bernoulli trials: $P_n = P($n trials result in even # of successes$)$

$E_n =$ n trials result in even # of successes
$E_n^c =$ n trials result in odd # of successes

$S_i =$ success on the ith trial

condition on S_1 (whether first trial is success)

$P_n = P(E_n) = P(E_n \mid S_1) P(S_1) + P(E_n \mid S_1^c) P(S_1^c)$

$P(E_n \mid S_1) = P(E_{n-1}^c)$ b/c need odd # of successes in next $n-1$ trials

$P(E_n \mid S_1^c) = P(E_{n-1})$ b/c need even # of successes in $n-1$ trials

So $P_n = (1-P_{n-1})p + P_{n-1}(1-p)$

Claim: $P_n = \dfrac{1+(1-2p)^n}{2}$

Base case:

$P_1 = P(S_1^c) = (1-p) = \dfrac{1+(1-2p)^1}{2}$ ✓

Induction step: Suppose $P_{n-1} = \dfrac{1+(1-2p)^{n-1}}{2}$

$P_n = p(1-P_{n-1}) + (1-p)P_{n-1} = p + P_{n-1} - 2pP_{n-1}$

$= p + \dfrac{1+(1-2p)^{n-1}}{2} - 2p\dfrac{1+(1-2p)^{n-1}}{2}$

$= \frac{1}{2}\left[2p+1+(1-2p)^{n-1} - 2p\left(1+(1-2p)^{n-1}\right)\right]$

$= \frac{1}{2}\left[\cancel{2p}+1+(1-2p)^{n-1} \cancel{-2p} - 2p(1-2p)^{n-1}\right]$

$= \frac{1}{2}\left[1+(1-2p)(1-2p)^{n-1}\right] = \frac{1}{2}\left[1+(1-2p)^n\right]$ ✓

QED.

3.18 E_n = no run of 3 consecutive Heads in n coin flips

$Q_n = P(E_n)$

Clearly $Q_0 = Q_1 = Q_2 = 1$

Our sample space may be broken into 4 parts, depending on how the sequence starts

$F_1 = \{T \cdots\}$ $\qquad P(F_1) = 1/2$

$F_2 = \{HT \cdots\}$ $\qquad P(F_2) = 1/4$

$F_3 = \{HHT \cdots\}$ $\qquad P(F_3) = 1/8$

$F_4 = \{HHH \cdots\}$ $\qquad P(F_4) = 1/8$

$$Q_n = P(E_n|F_1)P(F_1) + P(E_n|F_2)P(F_2) + P(E_n|F_3)P(F_3) + P(E_n|F_4)P(F_4)$$

$P(E_n|F_1) = P($ no 3 consecutive H on $n-1$ trials $) = Q_{n-1}$

$P(E_n|F_2) = P($ no 3 consecutive H on $n-2$ trials $) = Q_{n-2}$

$P(E_n|F_3) = Q_{n-3}$ (similarly)

$P(E_n|F_4) = 0$ since $F_3 \Rightarrow$ 3 consecutive heads

so $Q_n = \frac{1}{2}Q_{n-1} + \frac{1}{4}Q_{n-2} + \frac{1}{8}Q_{n-3}$

$Q_0 = Q_1 = Q_2 = 1$

$Q_3 = \frac{7}{8}$ $\qquad Q_4 = \frac{1}{2}\frac{7}{8} + \frac{1}{4} + \frac{1}{8} = \frac{13}{16}$

$Q_5 = \frac{1}{2}\frac{13}{16} + \frac{1}{4}\frac{7}{8} + \frac{1}{8} = \frac{24}{32}$

$Q_6 = \frac{1}{2}\frac{24}{32} + \frac{1}{4}\frac{13}{16} + \frac{1}{8}\frac{7}{8} = \frac{44}{64}$

$Q_7 = \frac{1}{2}\frac{44}{64} + \frac{1}{4}\frac{24}{32} + \frac{1}{8}\frac{13}{16} = \frac{81}{128}$

$Q_8 = \frac{1}{2}\frac{81}{128} + \frac{1}{4}\frac{44}{64} + \frac{1}{8}\frac{24}{32} = \frac{149}{256}$

Solutions 5

Ch 3 p. 113

3.26 (5.11) is $P(E_1|E_2F) = P(E_1|F)$

(5.12) is $P(E_1E_2|F) = P(E_1|F)P(E_2|F)$

These equations are equivalent

Suppose (5.11) holds: multiply both sides by $P(E_2|F)$

$$P(E_1|F)P(E_2|F) \overset{(5.11)}{=} P(E_1|E_2F)P(E_2|F)$$

$$= \frac{P(E_1E_2F)}{P(E_2F)}\frac{P(E_2F)}{P(F)} \quad \text{by definition of conditional Prob.}$$

$$= \frac{P(E_1E_2F)}{P(F)} = P(E_1E_2|F)$$

So (5.12) holds as well

Supposing (5.12) holds, divide both sides by $P(E_2|F)$ and use a similar argument to obtain (5.11)

Alternatively: $Q(E) = P(E|F)$ then Q is a probability

(5.11) $Q(E_1|E_2) = Q(E_1)$

(5.12) $Q(E_1E_2) = Q(E_1)Q(E_2)$

$\Bigg\}$ Know these are equivalent notions of independence.

3.28 Prove or give counterexample:

if E_1 and E_2 are independent, then they are conditionally independent given F.

Flip coin 2 times: E_1 = first is H
E_2 = second is H

$F = E_1 \cup E_2$ = first or second is H.

$P(E_1) = 1/2 \quad P(E_2) = 1/2 \quad P(F) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$

Then E_1 and E_2 are independent, BUT:

$$P(E_1 | F) = \frac{P(E_1(E_1 \cup E_2))}{P(F)} = \frac{P(E_1)}{P(F)} = \frac{1/2}{3/4} = \frac{2}{3}$$

$P(E_2 | F) = 2/3$ as well

$$P(E_1 E_2 | F) = \frac{P(E_1 E_2(E_1 \cup E_2))}{P(F)} = \frac{P(E_1 E_2)}{P(F)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Cond. Indep. means: $P(E_1 E_2 | F) = P(E_1 | F) P(E_2 | F)$

$\frac{1}{3} \neq \left(\frac{2}{3}\right) \cdot \left(\frac{2}{3}\right)$ so E_1 and E_2 are not conditionally independent, given F.

Recall:

3.29 C_i = ith coin selected

F_n = first n flips are Heads

Let H_m = next m flips are Heads

$$P(H_m|F_n) = \sum_{i=0}^{k} P(H_m|F_nC_i)\,P(C_i|F_n)$$

Just as before, we know $P(C_i|F_n) = \dfrac{(i/k)^n}{\sum_{j=0}^{k}(j/k)^n}$

Now $P(H_m|F_nC_i) = P(H_m|C_i) = (i/k)^m$,

since flips are conditionally independent, given C_i

$$\text{So } P(H_m|F_n) = \sum_{i=0}^{k} (i/k)^m \frac{(i/k)^n}{\sum_{j=0}^{k}(j/k)^n}$$

$$= \frac{\sum_{i=0}^{k}(i/k)^{n+m}}{\sum_{j=0}^{k}(j/k)^n} = \frac{\frac{1}{k}\sum_{i=0}^{k}(i/k)^{n+m}}{\frac{1}{k}\sum_{j=0}^{k}(j/k)^n}$$

As $k \to \infty$, $\frac{1}{k}\sum_{i=0}^{k}(i/k)^{n+m} \to \int_0^1 x^{n+m}\,dx = \frac{1}{n+m+1}$

$\frac{1}{k}\sum_{j=0}^{k}(j/k)^n \to \int_0^1 x^n\,dx = \frac{1}{n+1}$

So $P(H_m|F_n) \to \dfrac{n+1}{n+m+1}$

Ch 4 Problems pp. 172 – 173

4.1 Two balls chosen from

		win
8	white	$-\$1$
4	black	$+\$2$
2	orange	0

X = winnings. Possible Values: $-2, -1, 0, 1, 2, 4$

It is impossible to win exactly 3 if we draw two balls

There are $\binom{14}{2}$ possible outcomes

$$P(X=-2) = \frac{\binom{8}{2}}{\binom{14}{2}}$$ both white

$$P(X=-1) = \frac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}}$$ one white, one orange

$$P(X=0) = \frac{\binom{2}{2}}{\binom{14}{2}}$$ both orange

$$P(X=1) = \frac{\binom{8}{1}\binom{4}{1}}{\binom{14}{2}}$$ one white and one black

$$P(X=2) = \frac{\binom{4}{1}\binom{2}{1}}{\binom{14}{2}}$$ one black and one orange

$$P(X=4) = \frac{\binom{4}{2}}{\binom{14}{2}}$$ both black

4.3 Three dice rolled $6^3 = 216$ equally likely outcomes

X = sum of three dice

X can take any of the values 3 – 18

Value	Outcomes	Prob
3	(1,1,1)	1/216
4	(1,1,2) in 3 orders	3/216
5	(1,1,3) in 3 ways (1,2,2) in 3 ways	6/216
6	(1,1,4) in 3 ways (1,2,3) in 6 ways (2,2,2) in 1 way	10/216
7	(1,1,5) in 3 (1,2,4) in 6 (1,3,3) in 3 (2,2,3) in 3	15/216
8	(1,1,6) in 3 (1,2,5) in 6 (1,3,4) in 6 (2,2,4) in 3 (2,3,3) in 3	21/216

9	(1, 2, 6) in 6		
	(1, 3, 5) in 6		25/216
	(1, 4, 4) in 3		
	(2, 2, 5) in 3		
	(2, 3, 4) in 6		
	(3, 3, 3) in 1		
10	(1, 3, 6) in 6		
	(1, 4, 5) in 6		27/216
	(2, 2, 6) in 3		
	(2, 3, 5) in 6		
	(2, 4, 4) in 3		
	(3, 3, 4) in 3		
11	(1, 4, 6) in 6		
	(1, 5, 5) in 3		
	(2, 3, 6) in 6		27/216
	(2, 4, 5) in 6		
	(3, 3, 5) in 3		
	(3, 4, 4) in 3		
12	(1, 5, 6) in 6		
	(2, 4, 6) in 6		
	(2, 5, 5) in 3		25/216
	(3, 3, 6) in 3		
	(3, 4, 5) in 6		
	(4, 4, 4) in 1		

13	(1, 6, 6)	in 3	
	(2, 5, 6)	in 6	
	(3, 4, 6)	in 6	21/216
	(3, 5, 5)	in 3	
	(4, 4, 5)	in 3	
14	(2, 6, 6)	in 3	
	(3, 5, 6)	in 6	15/216
	(4, 4, 6)	in 3	
	(4, 5, 5)	in 3	
15	(3, 6, 6)	in 3	10/216
	(4, 5, 6)	in 6	
	(5, 5, 5)	in 1	
16	(4, 6, 6)	in 3	6/216
	(5, 5, 6)	in 3	
17	(5, 6, 6)	in 3	3/216
18	(6, 6, 6)	in 1	1/216

Note symmetry : $P(X = n) = P(X = 21 - n)$

Check: $(1+3+6+10+15+21+25+27) \times 2$

$= 108 \times 2 = 216$ So Probs add up to 1.

4.5 Coin is tossed n times

$X =$ # of heads $-$ # tails

Possible values: All heads : $X = n$

one tail : $X = (n-1) - 1 = n-2$

two tails : $X = (n-2) - 2 = n-4$

In general $X = n - 2(\#\text{ tails})$ so

$X = n, n-2, n-4, \dots, -n+4, -n+2, -n$

If we list possible values in decreasing order.

4.6 Assume $n=3$ and coin is fair.

So $X = -3, -1, 1, 3$ are possible

$P(X=-3) = P(\{TTT\}) = 1/8$

$P(X=-1) = P(\{HTT, THT, TTH\}) = 3/8$

$P(X=1) = P(\{HHT, HTH, THH\}) = 3/8$

$P(X=3) = P(\{HHH\}) = 1/8$

Solutions 6

Ch 4 problems

4.13 $P(\text{first sale}) = .3$ Each sale is equally likely to be \$1000 or \$500.

$P(\text{second sale}) = .6$

$$X = 0,\ 500,\ 1000,\ 1500,\ 2000$$

$X = 0 \iff$ neither sale is made $P = (1-.3)(1-.6)$ since sales independent

$$p(0) = P(X=0) = (.7)(.4) = .28$$

$X = 500 \iff$ exactly one sale of cheap version

$$\underset{\uparrow\ \text{first sale}}{(.3)}\underset{\uparrow\ \text{cheap version}}{\left(\tfrac{1}{2}\right)}\underset{\uparrow\ \text{not second sale}}{(1-.6)} + (1-.3)\underset{\uparrow\ \text{second sale}}{(.6)}\tfrac{1}{2}$$

$$p(500) = (.3)(.5)(.4) + (.7)(.6)(.5) = .06 + .21 = .27$$

$X = 1000 \iff$ one sale of expensive version, or both sales of cheap version

$$p(1000) = \underbrace{(.3)\left(\tfrac{1}{2}\right)(1-.6)}_{\text{first sale expensive}} + \underbrace{(1-.3)(.6)\left(\tfrac{1}{2}\right)}_{\text{second sale expensive}} + \underbrace{(.3)\left(\tfrac{1}{2}\right)(.6)\left(\tfrac{1}{2}\right)}_{\text{both sales cheap}}$$

$$= .06 + .21 + .045 = .315$$

$X = 1500$: both sales, one expensive, one cheap

$$p(1500) = \underbrace{(.3)\left(\tfrac{1}{2}\right)(.6)\left(\tfrac{1}{2}\right)}_{\text{first sale expensive}} + \underbrace{(.3)\left(\tfrac{1}{2}\right)(.6)\left(\tfrac{1}{2}\right)}_{\text{second sale expensive}}$$

$$= 2(.045) = .09$$

$X = 2000$ both sales expensive:

$$p(2000) = (.3)\left(\tfrac{1}{2}\right)(.6)\left(\tfrac{1}{2}\right) = .045$$

CHECK:

$.28 + .27 + .315 + .09 + .045 = 1$ ✓

4.18 Four indep coin flips $X = \#$ of heads

$P(X=0) = \left(\tfrac{1}{2}\right)^4 = \tfrac{1}{16}$ $\quad$ $P(X=1) = 4\left(\tfrac{1}{2}\right)^4 = \tfrac{1}{4}$

$P(X=2) = \binom{4}{2}\left(\tfrac{1}{2}\right)^4 = 6\,\tfrac{1}{16} = \tfrac{3}{8}$ $\quad$ $P(X=3) = \binom{4}{3}\left(\tfrac{1}{2}\right)^4 = 4\,\tfrac{1}{16} = \tfrac{1}{4}$

$P(X=4) = \left(\tfrac{1}{2}\right)^4 = \tfrac{1}{16}$

Let $p(a)$ denote the probability mass function of $X-2$

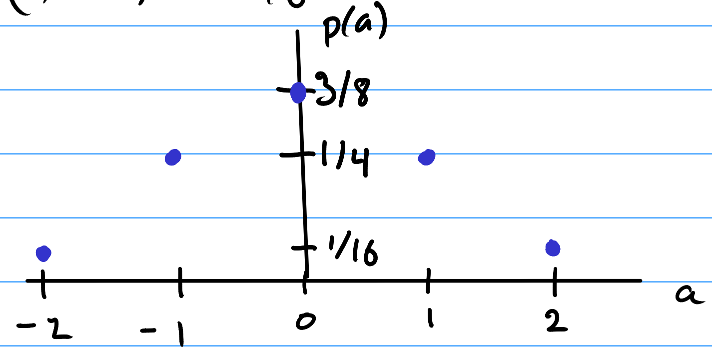
$$p(a) = P(X-2=a) = P(X=a+2)$$

$p(-2) = P(X=0) = 1/16$ $\quad$ $p(-1) = P(X=1) = 1/4$

$p(0) = P(X=2) = 3/8$ $\quad$ $p(1) = P(X=3) = 1/4$

$p(2) = P(X=4) = 1/16$

plot:



4.19 The distribution function of X is given by

$$F(b) = \begin{cases} 0 & b < 0 \\ 1/2 & 0 \le b < 1 \\ 3/5 & 1 \le b < 2 \\ 4/5 & 2 \le b < 3 \\ 9/10 & 3 \le b < 3.5 \\ 1 & 3.5 \le b \end{cases}$$

The probability mass function of X measures the jumps in $F(b)$.

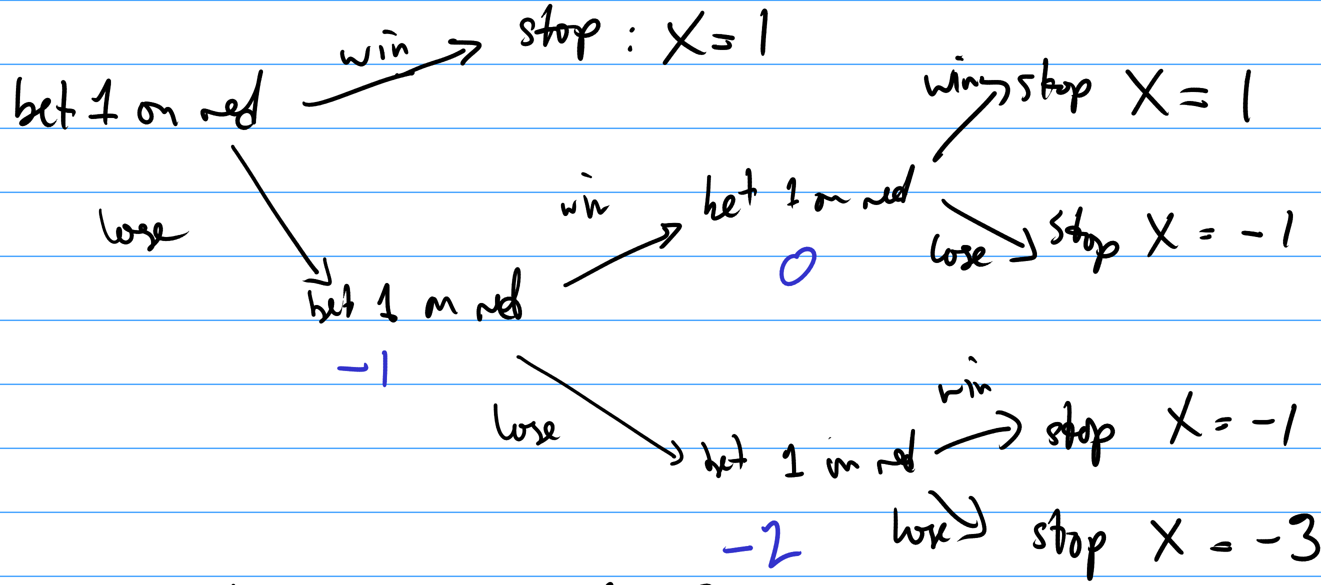
$p(0) = 1/2$, $p(1) = 3/5 - 1/2 = 1/10$,

$p(2) = 4/5 - 3/5 = 1/5$, $p(3) = 9/10 - 4/5 = 1/10$,

$p(3.5) = 1 - 9/10 = 1/10$,

$$p(b) = \begin{cases} 1/2 & b = 0 \\ 1/10 & b = 1 \\ 1/5 & b = 2 \\ 1/10 & b = 3 \\ 1/10 & b = 3.5 \\ 0 & \text{otherwise} \end{cases}$$

Roulette strategy: X = winnings



(a) Possible values: $X = 1, -1, -3$

$$P(X>0) = P(X=1) = P(\text{win}) + P(\text{lose, win, win})$$

$$= \frac{18}{38} + \frac{20}{38}\cdot\frac{18}{38}\cdot\frac{18}{38} = .474 + (.526)(.474)^2$$

$$.225$$

$$\approx .592$$

(b) The probability of winning something is greater than ½ so we are more likely to win than lose. But when we win, we get at most 1 dollar, while when we lose, we can lose up to 3 dollars. The expected value will tell the whole story.

(c) $$P(X=-1) = \left(\frac{20}{38}\right)\left(\frac{18}{38}\right)\left(\frac{20}{38}\right) + \left(\frac{20}{38}\right)\left(\frac{20}{38}\right)\left(\frac{18}{38}\right)$$

$$= \frac{14400}{54872} \approx .262$$

$$P(X=-3) = \left(\frac{20}{38}\right)\left(\frac{20}{38}\right)\left(\frac{20}{38}\right) = \frac{8000}{54872} \approx .146$$

$$E[X] \approx 1(.592) + (-1)(.262) + (-3)(.146)$$

$$= .592 - .262 - .438 = -0.108$$

The expected value is negative, so we will lose money in the long run.

4.28 3 items chosen from 20. 4 are defective

X = # of defective items in sample

$$P(X=0) = \binom{16}{3} \Big/ \binom{20}{3}$$
$$P(X=1) = \binom{4}{1}\binom{16}{2} \Big/ \binom{20}{3}$$
$$P(X=2) = \binom{4}{2}\binom{16}{1} \Big/ \binom{20}{3}$$
$$P(X=3) = \binom{4}{3} \Big/ \binom{20}{3}$$

$$E[X] = \left[0\binom{16}{3} + 1\binom{4}{1}\binom{16}{2} + 2\binom{4}{2}\binom{16}{1} + 3\binom{4}{3}\right] \Big/ \binom{20}{3}$$

$$= [4 \cdot 120 + 2 \cdot 6 \cdot 16 + 3 \cdot 4] \Big/ \left[\frac{20 \cdot 19 \cdot 18}{3 \cdot 2}\right]$$

$$= [480 + 192 + 12] / 1140 = 684/1140 = .6$$

Ch.4 Theoretical exercises:

4.2 Suppose X has cumulative distribution function F

So $F(x) = P(X \leq x)$

What is distribution function of e^X ?

Let $G(y) = P(e^X \le y)$ be the distribution function of e^X

$$e^X \le y \iff X \le \log y$$

In this equivalence, we use the fact that log is a monotonically increasing function

Thus $G(y) = P(e^X \le y) = P(X \le \log y) = F(\log y)$

So the distribution function of e^X is $F(\log y)$

4.3 If X has distribution function F, what is distribution function of $\alpha X + \beta$, where $\alpha \ne 0$?

Two cases are distinguished $\alpha > 0$ and $\alpha < 0$

Case $\alpha > 0$: $\quad \alpha X + \beta \le y \iff X \le \frac{1}{\alpha}(y - \beta)$

$$G(y) = P(\alpha X + \beta \le y) = P\left(X \le \frac{1}{\alpha}(y-\beta)\right) = F\left(\frac{1}{\alpha}(y-\beta)\right)$$

is the distribution function of $\alpha X + \beta$

Case $\alpha < 0$: $\quad \alpha X + \beta \le y \iff X \ge \frac{1}{\alpha}(y-\beta)$

$$P(\alpha X + \beta \le y) = P\left(X \ge \frac{1}{\alpha}(y-\beta)\right)$$

DIVISION BY α REVERSES INEQUALITY

$$= 1 - P\left(X < \frac{1}{\alpha}(y-\beta)\right) = 1 - \left[P\left(X \le \frac{1}{\alpha}(y-\beta)\right) - P\left(X = \frac{1}{\alpha}(y-\beta)\right)\right]$$

$= 1 - P\left(X \le \frac{1}{\alpha}(y-\beta)\right) + P\left(X = \frac{1}{\alpha}(y-\beta)\right)$

$= 1 - F\left(\frac{1}{\alpha}(y-\beta)\right) + p\left(\frac{1}{\alpha}(y-\beta)\right)$

where F is the cumulative distribution function of X
and p is the probability mass function of X

$$G(y) = 1 - F\left(\frac{1}{\alpha}(y-\beta)\right) + p\left(\frac{1}{\alpha}(y-\beta)\right)$$

If α is negative.

4.4 Let N be a random variable whose values are nonnegative integers.

Want to show $E[N] = \sum_{i=1}^{\infty} P\{N \ge i\}$

Using $P\{N \ge i\} = \sum_{k=i}^{\infty} P\{N = k\}$

gives $\sum_{i=1}^{\infty} P\{N \ge i\} = \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} P\{N = k\}$

this is a double sum over pairs (i, k) such that $k \ge i$

$$\begin{array}{c|cccc}
 & & k \rightarrow & & \\
i \downarrow & (1,1) & (1,2) & (1,3) & \cdots \\
 & & (2,2) & (2,3) & \cdots \\
 & & & (3,3) & \cdots \\
 & & & & \ddots
\end{array}$$

where k is summed over first

Summing over i first gives $\sum_{k=1}^{\infty} \sum_{i=1}^{k}$ (stuff)

Thus $\sum_{i=1}^{\infty} \sum_{k=i}^{\infty} P\{N=k\} = \sum_{k=1}^{\infty} \sum_{i=1}^{k} P\{N=k\}$

$= \sum_{k=1}^{\infty} k P\{N=k\} = E[N]$ by definition of expectation.

Technical point: reversing the order of summation is legitimate because the sum consists of positive terms:
The existance of $E[N]$ (which is tacitly assumed) implies convergence of the sum in the order $\sum_k \sum_i$ (stuff)

since the terms are positive, the convergence is absolute, and so the rearranged sum $\sum_i \sum_k$ (stuff) also converges

and has the same value.

4.7 X a random variable with $E[X] = \mu$ and $Var(X) = \sigma^2$

Find expectation and variance of $Y = \frac{X-\mu}{\sigma}$

$E[Y] = E\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}(E[X] - \mu)$ by Corollary 4.1

$= \frac{1}{\sigma}(\mu - \mu) = 0$

$Var(Y) = E[(Y - E[Y])^2] = E[Y^2]$

$E\left[\frac{(X-\mu)^2}{\sigma^2}\right] = \frac{1}{\sigma^2} E[(X-\mu)^2] = \frac{1}{\sigma^2} Var(X) = \frac{\sigma^2}{\sigma^2} = 1$

4.8 Find $Var(X)$ if $P(X=a)=p=1-P(X=b)$

$E[X] = a\,P(X=a) + b\,P(X=b) = ap + b(1-p)$

$= b + (a-b)p$

$Var(X) = E[X^2] - E[X]^2$

$E[X^2] = a^2p + b^2(1-p)$

$Var(X) = a^2p + b^2(1-p) - (b + (a-b)p)^2$

Solutions 7

4.30 A coin is flipped until a tail appears. Let Y denote the number of flips required. Then Y is a random variable and

$$P(Y=n) = P(\underbrace{HH\cdots H}_{n-1}T) = \left(\tfrac{1}{2}\right)^n$$

The payout in the St. Petersburg lottery is $X = 2^Y$

So $P(X = 2^n) = P(Y=n) = \left(\tfrac{1}{2}\right)^n$

Thus $E[X] = \sum_{n=1}^{\infty} 2^n \left(\tfrac{1}{2}\right)^n = \sum_{n=1}^{\infty} 1 = \infty$

(a) No. Let us compute

$$P(X \geq 2^k) = P(Y \geq k) = \sum_{n=k}^{\infty} \left(\tfrac{1}{2}\right)^n = \left(\tfrac{1}{2}\right)^k \frac{1}{1-\frac{1}{2}} = \left(\tfrac{1}{2}\right)^{k-1}$$

(geometric series)

In order to win at least 10^6, we would need to take $k = \lceil(\log_2 10^6)\rceil = \lceil 6 \log_2 10 \rceil = 20$

$$P(Y \geq 20) = \left(\tfrac{1}{2}\right)^{19} = \frac{1}{524288}$$

So it is extremely unlikely that we will make back our investment of 10^6 in one try

(b) (SOFT ANSWER) Yes you should play if you can play as much as you want and only have to settle up at the end.

Since $E[X] = \infty > 10^6$ we see that, Though on most games you will not win as much as you spent to play, there will be rare but extremely large wins which set to offset the losses

4.33 Buy papers at 10¢, sell at 15¢

Daily demand is binomial RV with parameters $n=10, p=\frac{1}{3}$

How many papers should he by to maximize his profit.

Let X be a random variable representing demand

so $P(X=i) = \binom{10}{i}\left(\frac{1}{3}\right)^i\left(\frac{2}{3}\right)^{n-i}$ $\quad i=0,1,\ldots,10$

If we buy k papers, we spend $10k$ (in ¢) and then we are able to sell up to k papers

So we actually sell $\min(X,k)$

and we make $15\min(X,k)$

So net profit is $15\min(X,k) - 10k$

Expected profit is $E\left[15\min(X,k) - 10k\right]$

$= 15\, E[\min(X,k)] - 10k$

$$E[\min(X,k)] = \sum_{i=0}^{k-1} i \binom{10}{i} \left(\tfrac{1}{3}\right)^i \left(\tfrac{2}{3}\right)^{n-i} + k \sum_{i=k}^{10} \binom{10}{i} \left(\tfrac{1}{3}\right)^i \left(\tfrac{2}{3}\right)^{n-i}$$

Using a computer to tabulate values

k	$15\, E[\min(X,k)] - 10k$
1	4.74
2	8.18
3	8.69 ←
4	5.30
5	-1.5
6	-10.35
7	-20.06
8	-30.01
9	-40.00
10	-50.00

So the expected profit is maximized when we buy 3 papers.

4.38 Suppose $E[X] = 1$ and $Var[X] = 5$

First find $E[X^2] = Var[X] + (E[X])^2 = 5 + (1)^2 = 6$

$\left(\text{using } Var[X] = E[X^2] - (E[X])^2\right)$

So: $E[(2+X)^2] = E[4 + 4X + X^2]$

$= 4 + 4E[X] + E[X^2] = 4 + 4(1) + (6) = 14$

$$Var[4+3X] = E[(4+3X)^2] - (E[4+3X])^2$$

$$E[4+3X] = 4+3E[X] = 4+3(1) = 7$$

$$E[(4+3X)^2] = E[16+24X+9X^2]$$

$$= 16+24E[X]+9E[X^2] = 16+24(1)+9(6) = 94$$

$$Var[4+3X] = 94-(7)^2 = 94-49 = 45$$

4.41 Man claims to have ESP. fair coin is flipped 10 times
Man gets 7 out of 10 correct. What is probability of doing at least this well with random guessing

Random guessing has probability of success $p=\frac{1}{2}$ on each flip. So # of correct guesses is a Bernoulli Random variable with $n=10$, $p=\frac{1}{2}$

$$P(X=7) = \binom{10}{7}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^3 = \frac{10!}{3!\,7!}\,\frac{1}{1024} = \frac{10\cdot 9\cdot 8}{3\cdot 2}\,\frac{1}{1024}$$

$$= \frac{10\cdot 3\cdot 4}{1024} = \frac{120}{1024} \approx .117$$

$$P(X=8) = \binom{10}{8}\frac{1}{1024} = \frac{10\cdot 9}{2}\cdot\frac{1}{1024} = \frac{45}{1024} \approx 0.044$$

$$P(X=9) = \binom{10}{9}\frac{1}{1024} = \frac{10}{1024} \approx 0.010$$

$$P(X=10) = \binom{10}{10}\frac{1}{1024} = \frac{1}{1024} \approx 0.001$$

$P(X\geq 7) = \frac{176}{1024} \approx .172 > \frac{1}{6}$ which isn't particularly small.

4.57 Suppose # of accidents on a highway each day is a Poisson Random Variable with $\lambda = 3$

(a) $P(X \geq 3) = 1 - P(X=0) - P(X=1) - P(X=2)$

$= 1 - e^{-3} - 3e^{-3} - \frac{(3)^2}{2} e^{-3} = 1 - 8.5e^{-3} \approx 0.577$

(b) $P(X \geq 3 \mid X \geq 1) = \dfrac{P(X \geq 3 \text{ and } X \geq 1)}{P(X \geq 1)} = \dfrac{P(X \geq 3)}{P(X \geq 1)}$

$P(X \geq 1) = 1 - P(X=0) = 1 - e^{-3} \approx 0.950$

so $P(X \geq 3 \mid X \geq 1) = 0.607$

4.59 play lottery 50 times $p = \frac{1}{100}$ in each

X = # of wins is binomial with parameters $n = 50$, $p = \frac{1}{100}$

We will approximate X by a Poisson RV. with $\lambda = np = \frac{1}{2}$

(a) $P(X \geq 1) = 1 - P(X=0) = 1 - e^{-\frac{1}{2}} \approx 0.393$

(b) $P(X=1) = \frac{1}{2} e^{-\frac{1}{2}} \approx 0.303$

(c) $P(X \geq 2) = 1 - P(X=0) - P(X=1) = 1 - e^{-\frac{1}{2}} - \frac{1}{2} e^{-\frac{1}{2}} \approx 0.090$

4.61 Probability of Full House $= 0.0014$

Let X = # of Full Houses dealt in 1000 hands

then X is binomial with parameters $n=1000$, $p=0.0014$

We approximate X by a Poisson R.V. with

$$\lambda = np = 1000 \cdot (0.0014) = 1.4$$

$$P(X \geq 2) = 1 - P(X=0) - P(X=1) = 1 - e^{-1.4} - 1.4\, e^{-1.4} \approx 0.408$$

Theoretical exercises

4.16 X is Poisson R.V with parameter λ

Show $P(X=i)$ increases monotonically and then decreases monotonically, and reaches maximum when $i = \lfloor \lambda \rfloor$

Consider $$\frac{P(X=i)}{P(X=i-1)} = \frac{\lambda^i e^{-\lambda}}{i!} \Big/ \frac{\lambda^{i-1} e^{-\lambda}}{(i-1)!} = \frac{\lambda}{i}$$

So if $i < \lambda$, then $1 < \frac{\lambda}{i}$, and so $P(X=i) > P(X=i-1)$

if $i > \lambda$, then $1 > \frac{\lambda}{i}$, and so $P(X=i) < P(X=i-1)$

So while $i < \lambda$, $P(X=i)$ increases monotonically
and when $i > \lambda$, $P(X=i)$ decreases monotonically

The maximum occurs for that i^* such that $P(X=i^*-1) \le P(X=i^*) > P(X=i^*+1)$

from the first inequality, we see that $i^* \le \lambda$
from the second we see $i^*+1 > \lambda$
So so $i^* \le \lambda < i^*+1$, and hence $i^* = \lfloor \lambda \rfloor$

4.19 Show that if X is a Poisson R.V. w/ parameter λ, then
$$E[X^n] = \lambda E[(X+1)^{n-1}]$$ and compute $E[X^3]$

Proof
$$E[X^n] = \sum_{i=0}^{\infty} i^n \frac{\lambda^i}{i!} e^{-\lambda} = \sum_{i=1}^{\infty} i^n \frac{\lambda^i}{i!} e^{-\lambda}$$

$$= \sum_{i=1}^{\infty} (i)^{n-1} \frac{\lambda^i}{(i-1)!} e^{-\lambda} = \lambda \sum_{i=1}^{\infty} (i)^{n-1} \frac{\lambda^{i-1}}{(i-1)!} e^{-\lambda}$$

$$= \lambda \sum_{j=0}^{\infty} (j+1)^{n-1} \frac{\lambda^j}{j!} e^{-\lambda}$$ Reindexing $j = i-1$

$$= \lambda \sum_{j=0}^{\infty} (j+1)^{n-1} P(X=j) = \lambda E[(X+1)^{n-1}]$$ QED.

So: $E[X] = \lambda E[(X+1)^0] = \lambda E[1] = \lambda \cdot 1 = \lambda$

$$E[X^2] = \lambda E[(X+1)^1] = \lambda(E[X]+1) = \lambda(\lambda+1) = \lambda^2 + \lambda$$

$$E[X^3] = \lambda E[(X+1)^2] = \lambda E[X^2 + 2X + 1]$$
$$= \lambda(E[X^2] + 2E[X] + 1) = \lambda[\lambda(\lambda+1) + 2\lambda + 1] = \lambda^3 + 3\lambda^2 + \lambda$$

Solutions 8

pp. 178–179

4.63 People enter casino at 1 per 2 minutes

Poisson process with rate parameter $\lambda = .5$ per min.

Between 12:00 and 12:05 = 5 min interval

X = Number who enter is Poisson distributed $\lambda t = (.5) \cdot 5 = 2.5$

(a) $P\{X=0\} = e^{-2.5}\frac{(2.5)^0}{0!} = e^{-2.5}$

(b) $P\{X \geq 4\} = 1 - P\{X<4\} = 1 - P\{X=0\} - P\{X=1\} - P\{X=2\} - P\{X=3\}$

$= 1 - e^{-2.5} - 2.5\,e^{-2.5} - \frac{(2.5)^2}{2}e^{-2.5} - \frac{(2.5)^3}{6}e^{-2.5}$

4.70 Coin starts on heads. It is flipped $N(t)$ times

where $N(t)$ is a poisson process with rate λ

$P\{N(t)=k\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$

What is $P\{$coin is showing heads at time $t\}$

(Note: coin is not necessarily fair : $P(\text{heads}) = p$)

$P(\text{coin showing heads at time } t)$

$= P(\text{coin showing heads at time } t \mid N(t)=0)\, P(N(t)=0)$

$+ P(\text{coin showing heads at time } t \mid N(t)>0)\, P(N(t)>0)$

Since coin starts on heads, $N(t)=0 \Rightarrow$ no flips
$\Rightarrow$ coin is showing heads at time t

$P(\text{head at time } t \mid N(t)=0) = 1$

If $N(t)>0$, then the coin has been flipped, but all that matters is the <u>last</u> flip, which has probability p of being heads

$P(\text{heads at time } t \mid N(t)>0) = p.$

Also $P(N(t)=0) = e^{-\lambda t}$ $\quad P(N(t)>0) = 1-e^{-\lambda t}$

so $P(\text{heads at time } t) = 1\cdot e^{-\lambda t} + p(1-e^{-\lambda t})$

$= p + (1-p)e^{-\lambda t}$

4.71 Roulette w/ 38 spaces. Smith bets on 1–12
has $p = \frac{12}{38}$ chance of winning each time

We have here Bernoulli trials with $p=\frac{12}{38}$

(a) $P(\text{Smith loses first 5 bets}) = P(\text{5 failures in a row})$

$$= (1-p)^5 = \left(1-\frac{12}{38}\right)^5 = \left(\frac{26}{38}\right)^5$$

(b) $P(\text{first win occurs on 4th bet}) = (1-p)^3 p$

$$= \left(1-\frac{12}{38}\right)^3\left(\frac{12}{38}\right) = \left(\frac{26}{38}\right)^3\left(\frac{12}{38}\right)$$ (geometric probability distribution)

4.75 A fair coin $(p=\frac{1}{2})$ is flipped until heads occurs for 10th time $\quad X = \#$ of tails

Let Y = total # of flips. Since we get 10 heads and X tails,

$$Y = X + 10$$

Also Y is negative binomial RV w/ $r=10$, $p=\frac{1}{2}$

$$P\{X=k\} = P\{Y=k+10\} = \binom{k+10-1}{10-1}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{k+10-10}$$

$$= \binom{k+9}{9}\left(\frac{1}{2}\right)^{k+10}$$

4.78 Urn w/ 4 white and 4 black. Choose 4 balls if get 2 white, 2 black, stop. Otherwise try again

$$P(\text{2 white/2 black}) = \frac{\binom{4}{2}\binom{4}{2}}{\binom{8}{4}} = \frac{6\cdot \cancel{6}^{3}}{\frac{8\cdot\cancel{7}\cdot 6\cdot 5}{4\cdot 3\cdot 2}} = \frac{18}{35}$$ (hypergeometric)

Now we're doing Bernoulli trials w/ $p = \frac{18}{35}$

X = # selections to first success (2white/2black) is geometric

$$P\{X=n\} = (1-p)^{n-1}p = \left(1-\frac{18}{35}\right)^{n-1}\left(\frac{18}{35}\right) = \left(\frac{17}{35}\right)^{n-1}\left(\frac{18}{35}\right)$$

P. 182

4.27 X = geometric w/ parameter p.

Show $P\{X=n+k \mid X>n\} = P\{X=k\}$

$$P\{X=n+k\} = (1-p)^{n+k-1}p$$

$$P\{X>n\} = P\{\text{first } n \text{ trials are failures}\} = (1-p)^n$$

$$P\{X=n+k \mid X>n\} = \frac{P\{X=n+k\}}{P\{X>n\}} = \frac{(1-p)^{n+k-1}p}{(1-p)^n}$$

$$= (1-p)^{k-1}p$$

But $P\{X=k\} = (1-p)^{k-1}p$ by PMF of geometric RV.

Conceptual argument

$X>n$ means the first n trials are failures. If we are looking for the first success, we can forget about the first n trials and pretend we are starting anew. Then the probability of getting 1st success on $(n+k)$th trial (after a string of n failures) is like getting 1st success on k th trial, starting from the beginning.

so $P\{X=n+k \mid X>n\} = P\{X=k\}$

4.29 For hypergeometric RV., compute

$$P\{X=k+1\} / P\{X=k\}$$

$$P\{X=k\} = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}} \quad \text{hypergeometric PMF.}$$

$$\frac{P\{X=k+1\}}{P\{X=k\}} = \frac{\binom{m}{k+1}\binom{N-m}{n-k-1}}{\binom{N}{n}} \Big/ \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$$

$$= \frac{\binom{m}{k+1}\binom{N-m}{n-k-1}}{\cancel{\binom{N}{n}}} \cdot \frac{\cancel{\binom{N}{n}}}{\binom{m}{k}\binom{N-m}{n-k}}$$

$$= \frac{\binom{m}{k+1}}{\binom{m}{k}} \cdot \frac{\binom{N-m}{n-k-1}}{\binom{N-m}{n-k}}$$

Lemma: $\binom{a}{b+1} \Big/ \binom{a}{b} = \dfrac{a-b}{b+1}$

Proof

$$\frac{\binom{a}{b+1}}{\binom{a}{b}} = \frac{\dfrac{a!}{(a-b-1)!(b+1)!}}{\dfrac{a!}{(a-b)!\,b!}} = \frac{\cancel{a!}}{(a-b-1)!\,(b+1)!} \cdot \frac{(a-b)!\,b!}{\cancel{a!}}$$

$$= \frac{(a-b)!}{(a-b-1)!} \cdot \frac{b!}{(b+1)!} = a-b \cdot \frac{1}{b+1} = \frac{a-b}{b+1} \quad \blacksquare$$

Now $\dfrac{\binom{m}{k+1}}{\binom{m}{k}} = \dfrac{m-k}{k+1}$ (use $a=m$, $b=k$)

and $\dfrac{\binom{N-m}{n-k}}{\binom{N-m}{n-k-1}} = \dfrac{N-m-n+k+1}{n-k}$ (use $a=N-m$, $b=n-k-1$)

so $\dfrac{P\{X=k+1\}}{P\{X=k\}} = \dfrac{m-k}{k+1} \cdot \dfrac{n-k}{N-m-n+k+1}$

P. 224

5.1 X continuous RV PDF $f(x)$

$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) value of c?

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-1}^{1} c(1-x^2)\,dx = c\left[x - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= c\left[1 - \frac{1}{3} - \left(-1 - \frac{-1}{3}\right)\right] = c\left[\frac{2}{3} - \left(-\frac{2}{3}\right)\right] = c\,\frac{4}{3}$$

so $c = \frac{3}{4}$

Cumulative distribution function $F(a) = \int_{-\infty}^{a} f(x)\,dx$

If $a \leq -1$ $F(a) = \int_{-\infty}^{a} 0\,dx = 0$

If $-1 < a < 1$ $F(a) = \int_{-\infty}^{a} f(x)\,dx = \int_{-1}^{a} \frac{3}{4}(1-x^2)\,dx$

$$= \left[\frac{3}{4}\left(x - \frac{x^3}{3}\right)\right]_{-1}^{a} = \frac{3}{4}\left(a - \frac{a^3}{3}\right) - \frac{3}{4}\underbrace{\left(-1 - \frac{-1}{3}\right)}_{-\frac{2}{3}}$$

$$= \frac{3}{4}\left(a - \frac{a^3}{3}\right) + \frac{1}{2}$$

If $1 \leq a$ $F(a) = \int_{-\infty}^{a} f(x)\,dx = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} \frac{3}{4}(1-x^2)\,dx + \int_{1}^{a} 0\,dx = 1$

So $F(a) = \begin{cases} 0 & \text{if } a \leq -1 \\ \frac{3}{4}\left(a - \frac{a^3}{3}\right) + \frac{1}{2} & \text{if } -1 < a < 1 \\ 1 & \text{if } 1 \leq a \end{cases}$

Solutions 9

5.2 System functions for amount of time X

PDF is $f(x) = \begin{cases} Cxe^{-x/2} & x > 0 \\ 0 & x \le 0 \end{cases}$

x is in months

what is $P(X \geq 5)$?

First find $\int xe^{-x/2}dx = -2xe^{-x/2} - \int(-2)e^{-x/2}dx$

(integration by parts)

$$= -2xe^{-x/2} - 4e^{-x/2} + \text{constant}$$

So $1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^{\infty} Cxe^{-x/2}dx = C\left[-2xe^{-x/2} - 4e^{-x/2}\right]_0^{\infty}$

$$= C\left[0 - \left(-2\cdot 0\cdot e^{-0/2} - 4e^{-0/2}\right)\right] = C\cdot 4$$

so $C = \frac{1}{4}$

$P(X \geq 5) = \int_5^{\infty} f(x)dx = \int_5^{\infty} \frac{1}{4}xe^{-x/2}dx$

$$= \frac{1}{4}\left[-2xe^{-x/2} - 4e^{-x/2}\right]_5^{\infty}$$

$$= \frac{1}{4}\left[0 - \left(-2\cdot 5\cdot e^{-5/2} - 4e^{-5/2}\right)\right] = \frac{1}{4}(14)e^{-5/2} = \frac{7}{2}e^{-5/2}$$

5.3 Consider $f(x) = \begin{cases} C(2x-x^3) & 0<x<\frac{5}{2} \\ 0 & \text{otherwise} \end{cases}$

$f(2) = C(2\cdot 2 - 2^3) = C(4-8) = C(-4)$

$f(1) = C(2\cdot 1 - 1^3) = C(2-1) = C\cdot 1$

Since the probability density function of a random variable must be nonnegative,

$0 \le f(2) = C(-4) \implies C \le 0$
$0 \le f(1) = C(1) \implies C \ge 0$
So $C=0$ is the only possibility.

But then $f(x)=0$ everywhere, and $\int_{-\infty}^{\infty} f(x)dx = 0$, which is impossible if f is a PDF

for $f(x) = \begin{cases} C(2x-x^2) & 0<x<\frac{5}{2} \\ 0 & \text{otherwise} \end{cases}$

we see $f(1) = C(2-1) = C(1) \implies C \ge 0$
$f(2.1) = C(4.2 - 4.41) = C(-.21) \implies C \le 0$

so $C=0$ but $\int_{-\infty}^{\infty} f(x)\,dx$ can't be zero

So neither function can be the PDF of a RV.

5.4 X = life time of device (in hours)

has PDF $f(x) = \begin{cases} \frac{10}{x^2} & x > 10 \\ 0 & x \leq 10 \end{cases}$

First compute $\int \frac{10}{x^2}\,dx = -\frac{10}{X} + \text{constant}$

(a) $P\{X > 20\} = \int_{20}^{\infty} \frac{10}{x^2}\,dx = \left[-\frac{10}{x}\right]_{20}^{\infty} = 0 - \left(-\frac{10}{20}\right) = \frac{1}{2}$

(b) Cumulative distribution function $F(a) = \int_{-\infty}^{a} f(x)\,dx$

$$\int_{-\infty}^{a} f(x)\,dx = \int_{10}^{a} \frac{10}{x^2}\,dx = \left[-\frac{10}{x}\right]_{10}^{a} = -\frac{10}{a} - \left(-\frac{10}{10}\right)$$

$$F(a) = 1 - \frac{10}{a}$$

(c) Given six such devices, what is probability at least 3 function for at least 15 hours?

Assume each of 6 devices functions/fails independently
Then we have Bernoulli trials with $n = 6$
and $p = P\{X > 15\}$:

$$p = P\{X \geq 15\} = 1 - P\{X \leq 15\} = 1 - F(15) = \frac{10}{15} = \frac{2}{3}$$

$$P(\text{exactly } k \text{ of } 6 \text{ last } 15 \text{ hours}) = \binom{6}{k} p^k (1-p)^{6-k}$$

$$= \binom{6}{k}\left(\frac{2}{3}\right)^k\left(\frac{1}{3}\right)^{6-k} = \binom{6}{k}\frac{2^k}{3^6}$$

$P(\geq 3 \text{ last at least 15}) = P(3) + P(4) + P(5) + P(6)$

$$= \binom{6}{3}\frac{2^3}{3^6} + \binom{6}{4}\frac{2^4}{3^6} + \binom{6}{5}\frac{2^5}{3^6} + \binom{6}{6}\frac{2^6}{3^6}$$

5.6 Compute $E[X]$:

(a) $f(x) = \begin{cases} \frac{1}{4} x e^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} \frac{1}{4} x^2 e^{-x/2}\,dx$$

integrate by parts $\quad u = x^2 \quad dv = e^{-x/2}dx$

$du = 2x\,dx \quad v = -2e^{-x/2}$

$$\int u\,dv = uv - \int v\,du$$

$$\int \frac{1}{4}x^2 e^{-x/2} dx = \frac{1}{4}x^2(-2e^{-x/2}) - \int \frac{1}{4} 2x(-2)e^{-x/2}\,dx$$

$$= -\frac{1}{2}x^2 e^{-x/2} + \int x e^{-x/2}\,dx$$

integrate by parts

$\alpha = x \quad d\beta = e^{-x/2}dx$

$d\alpha = dx \quad \beta = -2e^{-x/2}$

$$= -\frac{1}{2}x^2 e^{-x/2} + (-2xe^{-x/2}) - \int -2e^{-x/2}\,dx$$

$$= -\frac{1}{2}x^2 e^{-x/2} - 2xe^{-x/2} - 4e^{-x/2} + \text{constant.}$$

$$= \left(-\frac{1}{2}x^2 - 2x - 4\right)e^{-x/2}$$

Check $\frac{d}{dx}\left(-\frac{1}{2}x^2-2x-4\right)e^{-x/2}$

$= (-x-2)e^{-x/2} + \left(-\frac{1}{2}x^2-2x-4\right)\left(-\frac{1}{2}\right)e^{-x/2}$

$= \left(-x-2+\frac{1}{4}x^2+x+2\right)e^{-x/2} = \frac{1}{4}x^2e^{-x/2}$ ✓

$E[X] = \int_0^\infty \frac{1}{4}x^2e^{-x/2}\,dx = \left[\left(-\frac{1}{2}x^2-2x-4\right)e^{-x/2}\right]_0^\infty$

Now $\lim_{x\to\infty} x^n e^{-ax} = 0$ for any $n \geq 0$ & $a > 0$

(this may be proved using repeated applicati of L'Hospital's rule)

Thus $\lim_{x\to\infty}\left(-\frac{1}{2}x^2-2x-4\right)e^{-x/2} = 0$

$E[X] = 0 - \left(-\frac{1}{2}(0)^2-2(0)-4\right)e^{-0/2} = 4$

(b) $f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

Recall from 5.1 that $c = \frac{3}{4}$

So $E[X] = \int_{-\infty}^{\infty} xf(x)\,dx = \int_{-1}^{1} x\left(\frac{3}{4}\right)(1-x^2)\,dx$

$= \frac{3}{4}\int_{-1}^{1}(x-x^3)\,dx = \frac{3}{4}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_{-1}^{1}$

$= \frac{3}{4}\left[\left(\frac{1}{2}-\frac{1}{4}\right)-\left(\frac{(-1)^2}{2}-\frac{(-1)^4}{4}\right)\right] = \frac{3}{4}\left[\left(\frac{1}{2}-\frac{1}{4}\right)-\left(\frac{1}{2}-\frac{1}{4}\right)\right] = 0$

(c) $f(x) = \begin{cases} \frac{5}{x^2} & x > 5 \\ 0 & x \le 5 \end{cases}$

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_5^{\infty} x\left(\frac{5}{x^2}\right) dx = \int_5^{\infty} \frac{5}{x} dx$$

$$= \left[5 \ln(x)\right]_5^{\infty} = 5\left(\lim_{x\to\infty} \ln(x) - \ln(5)\right)$$

Now $\lim_{x\to\infty} \ln(x) = \infty$ so this integral diverges

$E[X] = \infty$.

5.7 $f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ and $E[X] = \frac{3}{5}$

Thus we have two unknowns: a, b
and two equations:

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \qquad \int_{-\infty}^{\infty} x f(x)\, dx = E[X] = \frac{3}{5}$$

$$1 = \int_0^1 (a + bx^2) dx = \left[ax + b\frac{x^3}{3}\right]_0^1 = a + \frac{b}{3}$$

$$\frac{3}{5} = \int_0^1 x(a+bx^2) dx = \int_0^1 (ax + bx^3) dx = \left[a\frac{x^2}{2} + b\frac{x^4}{4}\right]_0^1 = \frac{a}{2} + \frac{b}{4}$$

so $\begin{cases} a + \frac{b}{3} = 1 \Rightarrow 3a + b = 3 \\ \frac{a}{2} + \frac{b}{4} = \frac{3}{5} \Rightarrow 2a + b = \frac{12}{5} \end{cases}$ subtract $\Rightarrow a = 3 - \frac{12}{5}$

$a = \frac{15}{5} - \frac{12}{5} = \frac{3}{5}$

so $b = \frac{12}{5} - 2a = \frac{12}{5} - \frac{6}{5} = \frac{6}{5}$

So $a = \frac{3}{5}$ $b = \frac{6}{5}$

$$f(x) = \begin{cases} \frac{3}{5} + \frac{6}{5}x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

5.8 X = lifetime in hours of device. PDF is:

$f(x) = xe^{-x}$ $x \geq 0$

Expected lifetime $= E[X] = \int_0^\infty x f(x)dx = \int_0^\infty x^2 e^{-x} dx$

[This is similar to 5.6(a)]

$$\int x^2 e^{-x} dx = -x^2 e^{-x} - \int 2x(-e^{-x})dx$$

$$= -x^2 e^{-x} + 2\int x e^{-x} dx$$

$$= -x^2 e^{-x} + 2\left[-xe^{-x} - \int -e^{-x} dx\right]$$

$$= -x^2 e^{-x} - 2xe^{-x} + 2\int e^{-x} dx$$

$$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} = -(x^2+2x+2)e^{-x}$$

Check: $\frac{d}{dx}\left[-(x^2+2x+2)e^{-x}\right] =$

$= -(2x+2)e^{-x} - (x^2+2x+2)(-e^{-x}) = x^2 e^{-x}$ ✓

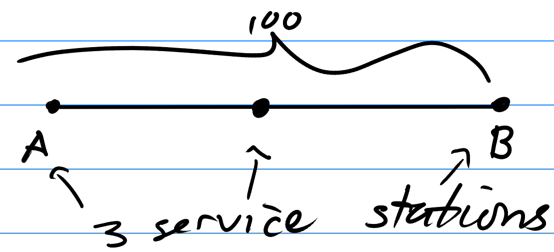
So $E[X] = \int_0^\infty x^2 e^{-x} dx = \left[-(x^2+2x+2)e^{-x}\right]_0^\infty$

$$= \left[\lim_{x \to \infty} -(x^2+2x+2) e^{-x} \right] + 2 = 0+2 = 2$$

(since $\lim_{x \to \infty} x^n e^{-x} = 0$ for $n \geq 0$)

Solutions 10

5.12



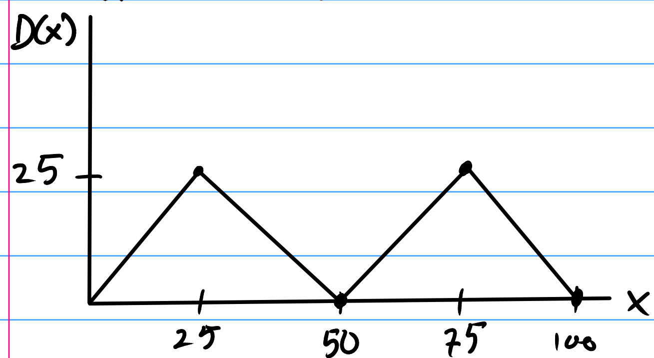
X = location of breakdown is uniform on $(0, 100)$

PDF $f(x) = \begin{cases} \frac{1}{100} & 0 < x < 100 \\ 0 & \text{otherwise} \end{cases}$

Let $D(X)$ be the distance from location of breakdown

$$D(X) = \begin{cases} X, & 0 < X \leq 25 \\ 50 - X, & 25 < X \leq 50 \\ X - 50, & 50 < X \leq 75 \\ 100 - X, & 75 < X < 100 \end{cases}$$

Which looks like



$$E[D(X)] = \int_0^{100} D(x) f(x)\,dx = \int_0^{100} D(x) \frac{1}{100}\,dx = \frac{1}{100} \int_0^{100} D(x)\,dx$$

$= \frac{1}{100}$ (Area under the graph of $D(x)$)

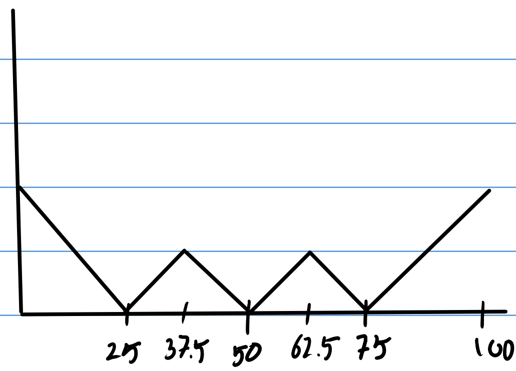
$= \frac{1}{100}(4)\left(\frac{1}{2}\, 25 \cdot 25\right)$ (break up into 4 triangles)

$= \frac{1}{2} \cdot 25 = 12.5$

If stations are located at 25, 50, and 75,

$$D(x) = \begin{cases} 25 - x & 0 < x \le 25 \\ x - 25 & 25 < x \le 37.5 \\ 50 - x & 37.5 < x \le 50 \\ x - 50 & 50 < x \le 62.5 \\ 75 - x & 62.5 < x \le 75 \\ x - 75 & 75 < x < 100 \end{cases}$$

which looks like



$$E[D(X)] = \int_0^{100} D(x) \frac{1}{100} dx$$

$$= \frac{1}{100} (\text{Area under curve})$$

$$= \frac{1}{100}\left[4 \cdot \left(\frac{1}{2}(12.5)^2\right) + 2\left(\frac{1}{2} 25^2\right)\right]$$

$$= \frac{1}{100}[312.5 + 625] = \frac{1}{100}[937.5] = 9.375$$

Since $9.375 < 12.5$, the second scheme is more efficient.

5.13 You arrive a bus stop at 10.
Arrival of bus is uniformly distributed between 10 and 10:30.

(a) What is probability we have to wait longer than 10 minutes

Wait longer than 10 minutes $\iff$ bus arrives between 10:10 and 10:30

$$P(10:10 < X < 10:30) = \frac{20 \text{ minutes}}{30 \text{ minutes}} = \frac{2}{3}$$

(b) If, at 10:15 bus has not arrived, find probability you must wait another 10 minutes

Wait another 10 minutes $\Leftrightarrow$ bus arrives between 10:25 and 10:30

$$P(10{:}25 < X < 10{:}30 \mid 10{:}15 < X) = \frac{P(10{:}25 < X < 10{:}30)}{P(10{:}15 < X)}$$

$$\frac{5/30}{15/30} = \frac{5}{15} = \frac{1}{3}.$$

5.15 X normal with $\mu = 10$ and $\sigma^2 = 36$

compute $\sigma = 6$

let Z denote standard normal RV.

NOTE: I USED THE TABLE ON P. 201 to look up $\Phi(x)$
CALCULATOR MAY GIVE SLIGHTLY DIFFERENT RESULTS

(a) $P\{X > 5\} = P\{\frac{X-10}{6} > \frac{5-10}{6}\} = P\{Z > \frac{-5}{6}\} = P\{Z < \frac{5}{6}\}$

$$= \Phi\left(\tfrac{5}{6}\right) \approx \Phi(.83) \approx .7967$$

(b) $P\{4 < X < 16\} = P\{\frac{4-10}{6} < Z < \frac{16-10}{6}\} = P\{-1 < Z < 1\}$

$$= \Phi(1) - \Phi(-1) = \Phi(1) - [1 - \Phi(1)] = 2\Phi(1) - 1$$

$$= 2(.8413) - 1 = .6826$$

(c) $P\{X < 8\} = P\{Z < \frac{8-10}{6}\} = P\{Z < -\frac{1}{3}\} = 1 - \Phi(\frac{1}{3})$

$$= 1 - \Phi(.33) = 1 - (.6293) = .3707$$

(d) $P\{X<20\} = P\{Z < \frac{20-10}{6}\} = P\{Z < \frac{5}{3}\}$

$= \Phi(\frac{5}{3}) \approx \Phi(1.66) = .9515$

(e) $P\{X>16\} = P\{Z > \frac{16-10}{6}\} = P\{Z>1\} = 1-\Phi(1)$

$= 1-.8413 = .1587$

5.16 X = annual rainfall is normal with $\mu = 40$, $\sigma = 4$
(X is in inches)

$P\{X>50\} = P\{\frac{X-40}{4} > \frac{50-40}{4}\} = P\{\frac{X-40}{4} > 2.5\}$

$= 1-\Phi(2.5) = 1-(.9938) = .0062$

The number of years it takes to have a year with over 50 inches of rain is a geometric random variable with $p = .0062$ (Assuming the rainfalls in different years are independent)

So $P\{$take over 10 years to get > 50 inches$\}$

$= (1-p)^{10} = (.9938)^{10} = .9397$

5.18 X is normal with $\mu = 5$ and unknown σ.
If we know $P\{X>9\} = .2$, then

$.2 = P\{\frac{X-5}{\sigma} > \frac{9-5}{\sigma}\} = P\{\frac{X-5}{\sigma} > \frac{4}{\sigma}\} = 1-\Phi(\frac{4}{\sigma})$

$\therefore \Phi(\frac{4}{\sigma}) = 1-.2 = .8$

Doing a reverse lookup in the table for Φ, we see

$$\frac{4}{\sigma} \approx .85 \implies \sigma \approx \frac{4}{.85} = 4.7$$

Thus $Var(X) = \sigma^2 \approx 22$

5.23 1000 independent rolls of a fair die
$X = \#$ of 6's rolled

Compute approximation to $P\{150 \le X \le 200\}$

X is binomial with $n = 1000$ $p = \frac{1}{6}$
$\mu = np = 166.6$
$\sigma^2 = np(1-p) = 138.8$ $\sigma = 11.79$

Apply DeMoivre-Laplace Limit theorem

$P\{150 \le X \le 200\} = P\{149.5 < X < 200.5\}$ continuity correction

$$\approx P\left\{\frac{149.5 - 166.6}{11.79} < Z < \frac{200.5 - 166.6}{11.79}\right\}$$

$$\approx P\{-1.450 < Z < 2.875\}$$

$$= \Phi(2.875) - \Phi(-1.450)$$

$$\approx .9244$$

Note: I used computer for some of these steps. Rounding and using table may give slightly different answer.

Second part: Assume 6 appears exactly 200 times

Find conditional probability that 5 appears less than 150 times.

FACT: Assuming 6 appears exactly 200 times, the number of 5's which appear is a binomial Random var. with $n=800$ and $p=\frac{1}{5}$

Quick intuitive argument: If 6 appears 200 times, we can set those 200 trials aside, and we have $n=800$ trials on which a 5 might appear. Since we know a 6 cannot appear on any of these 800 trials, the probability of getting a 5 is

$$P(5 \mid \text{not } 6) = \frac{P(5)}{P(\text{not } 6)} = \frac{1/6}{5/6} = \frac{1}{5}, \text{ so } p = \frac{1}{5}$$

Proof with formulas let $Y = \#$ of 5's $X = \#$ 6's

$$P(Y=k \mid X=200) = \frac{P(Y=k \;\&\; X=200)}{P(X=200)}$$

$$P(X=200) = \binom{1000}{200}\left(\frac{1}{6}\right)^{200}\left(\frac{5}{6}\right)^{800}$$

$$P(Y=k \;\&\; X=200) = \underbrace{\binom{1000}{200}}_{\text{where 6's go}}\underbrace{\binom{800}{k}}_{\text{where 5's go}}\underbrace{\left(\frac{1}{6}\right)^{200}}_{\text{200 6's}}\underbrace{\left(\frac{1}{6}\right)^{k}}_{k \text{ 5's}}\underbrace{\left(\frac{4}{6}\right)^{800-k}}_{800-k \text{ 1,2,3,4 s}}$$

$$P(Y=k \mid X=200) = \frac{\cancel{\binom{1000}{200}}\binom{800}{k}\cancel{\left(\frac{1}{6}\right)^{200}}\left(\frac{1}{6}\right)^{k}\left(\frac{4}{6}\right)^{800-k}}{\cancel{\binom{1000}{200}}\cancel{\left(\frac{1}{6}\right)^{200}}\left(\frac{5}{6}\right)^{800}}$$

$$= \binom{800}{k}\left(\frac{1}{6}\right)^{k}\left(\frac{4}{6}\right)^{800-k}\left(\frac{6}{5}\right)^{800} = \binom{800}{k} 1^{k} 4^{800-k} \cancel{\left(\frac{1}{6}\right)^{800}} \frac{\cancel{6^{800}}}{5^{800}}$$

$$= \binom{800}{k}\left(\frac{1}{5}\right)^{k}\left(\frac{4}{5}\right)^{800-k}$$

which is binomial with $n=800$ $p=\frac{1}{5}$

$P\{Y < 150 \mid X=200\}$ can be approximated

$np = 160$ $np(1-p) = 128$ $\sqrt{np(1-p)} = 8\sqrt{2} \approx 11.31$

$$P\{Y<150 \mid X=200\} = P\{Y < 149.5 \mid X=200\}$$

$$= P\left\{Z < \frac{149.5-160}{11.31}\right\}$$

$$= P\{Z < -.9284\} = \Phi(-.9284) \approx .1766$$

5.27 In 10000 tosses coin lands on heads 5800 times

Is it reasonable to conclude coin is unfair

Suppose coin is fair: then $X = \#$ heads is binomial with $n = 10000$, $p=\frac{1}{2}$, $\mu = np = 5000$, $\sigma^2 = np(1-p) = 2500$

$\sigma = \sqrt{2500} = 50$

since $\mu = 5000$, $\sigma = 50$, $5800 = \mu + 16\sigma$

So 5800 heads represents at deviation from mean of 16 standard deviations

So $P\{X \geq 5800\} = P\{Z \geq 16\} = 1 - \Phi(16)$,
and this number is essentially zero.

In fact even:

$P\{X \geq 5100\} = P\{Z \geq 2\} \approx .02275 = 2.275\%$ is quite small

$P\{X \geq 5400\} = P\{Z \geq 8\} \approx 6.66 \times 10^{-16}$ is very veryvery small.

$P\{X \geq 5800\} = P\{Z \geq 16\}$ is so small it is outside the limits of precision on my computer.

So it is nearly certain the coin is <u>not</u> fair.

P. 227 – 228

5.2 Statement: $E[Y] = \int_0^\infty P\{Y > y\}dy - \int_0^\infty P\{Y < -y\}dy$

<u>Step 1</u>: $\int_0^\infty P\{Y > y\}dy = \int_0^\infty x f_Y(x)\,dx$

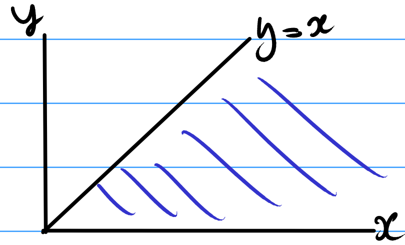
(this is the same proof as in class)

$$P\{Y > y\} = \int_y^\infty f_Y(x)\,dx$$

So $\int_0^\infty P\{Y>y\}dy = \int_0^\infty \int_y^\infty f_Y(x)\,dx\,dy$

the integral is over the region $\{(x,y): 0<y<\infty,\ y<x<\infty\}$

Picture:



$= \{(x,y): 0<x<\infty,\ 0<y<x\}$

Change order of integration: $= \int_0^\infty \int_0^x f_Y(x)\,dy\,dx$

$= \int_0^\infty \left[f_Y(x)\,y\right]_{y=0}^{y=x} dx = \int_0^\infty x f_Y(x)\,dx$ Done with Step 1.

Step 2 $\int_0^\infty P\{Y<-y\}dy = \int_{-\infty}^0 x f_Y(x)\,dx$

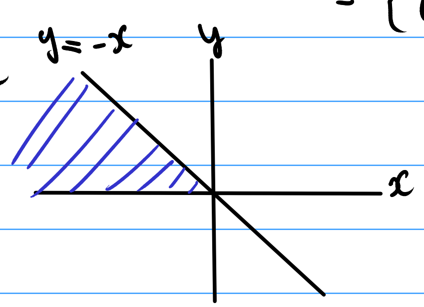
$P\{Y<-y\} = \int_{-\infty}^{-y} f_Y(x)\,dx$

So $\int_0^\infty P\{Y<-y\}dy = \int_0^\infty \int_{-\infty}^{-y} f_Y(x)\,dx\,dy$

Region of integration $\{(x,y): 0<y<\infty,\ -\infty<x<-y\}$

$= \{(x,y): -\infty<x<0,\ 0<y<-x\}$

Picture



Switch order of integration

$$= \int_{-\infty}^{0} \int_{0}^{-x} f_Y(x)\, dy\, dx = \int_{-\infty}^{0} \Big[y f_Y(x) \Big]_{y=0}^{y=-x} dx$$

$$= \int_{-\infty}^{0} (-x) f_Y(x) dx = - \int_{-\infty}^{0} f_Y(x) dx$$ Done with Step 2

Finally,

Step 3: $E[Y] = \int_{-\infty}^{\infty} x f_Y(x) dx$ by definition

$$= \int_{-\infty}^{0} x f_Y(x) dx + \int_{0}^{\infty} x f_Y(x) dx$$

$$= - \int_{0}^{\infty} P\{Y < -y\} dy + \int_{0}^{\infty} P\{Y > y\} dy$$ QED

5.3 Statement: Let $Y = g(X)$ then

$$E[Y] = E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

Using 5.2, we have

$$E[Y] = \int_{0}^{\infty} P\{Y > y\} dy - \int_{0}^{\infty} P\{Y < -y\} dy$$

$$= \int_{0}^{\infty} P\{g(X) > y\} dy - \int_{0}^{\infty} P\{g(X) < -y\} dy$$

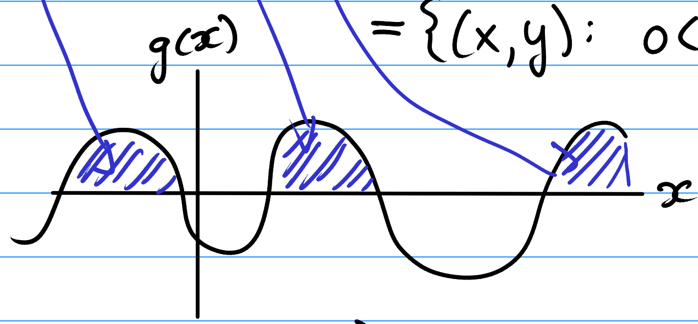
Analyze $\int_0^\infty P\{g(X)>y\}\,dy$

$$\left(P\{g(X)>y\} = \int_{\substack{x \text{ such} \\ \text{that } g(x)>y}} f_X(x)\,dx \right)$$

$$= \int_0^\infty \int_{\substack{x \text{ such that} \\ g(x)>y}} f_X(x)\,dx\,dy$$

Region of integration: $\{(x,y):\ 0<y<\infty,\ \ y<g(x)\}$

$= \{(x,y):\ 0<g(x),\ \ 0<y<g(x)\}$



$$= \int_{\substack{x \text{ such} \\ \text{that } g(x)>0}} \int_0^{g(x)} f_X(x)\,dy\,dx$$

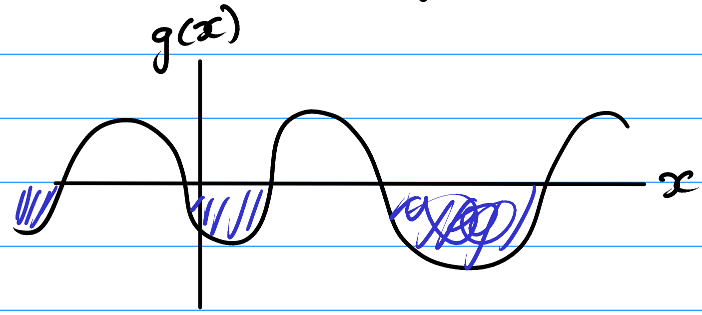
$$= \int_{\substack{x \text{ such that} \\ g(x)>0}} g(x) f_X(x)\,dx$$

Analyze $-\int_0^\infty P\{g(X) < -y\}\,dy$

$$= -\int_0^\infty \int_{\substack{x \text{ such that} \\ g(x) < -y}} f_X(x)\,dx\,dy$$

Region : $\{(x,y) : 0 < y < \infty, \quad g(x) < -y\}$

$\{(x,y) : g(x) < 0, \quad 0 < y < -g(x)\}$



$$= -\int_{\substack{x \text{ such that} \\ g(x) < 0}} \int_0^{-g(x)} f_X(x)\,dx = -\int_{\substack{x \text{ such that} \\ g(x) < 0}} -g(x) f_X(x)\,dx$$

$$= \int_{\substack{x \text{ such that} \\ g(x) < 0}} g(x) f_X(x)\,dx$$

$$E[g(X)] = \int_{\substack{x \text{ such that} \\ g(x) > 0}} g(x) f_X(x)\,dx + \int_{\substack{x \text{ such that} \\ g(x) < 0}} g(x) f_X(x)\,dx$$

Now $\int_{\substack{x \text{ such that} \\ g(x) = 0}} g(x) f_X(x)\,dx = 0$

So we can neglect the set where $g(x)=0$, and we get

$$E[g(X)] = \int_{\text{all } x} g(x) f_X(x)\,dx = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

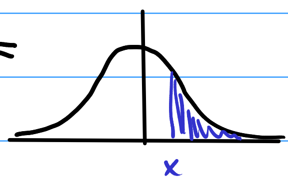
$(g(x)>0, g(x)<0, \text{ or } g(x)=0)$

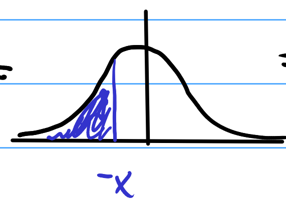
QED.

5.9 Z standard normal random variable, and $x>0$

(a) $P\{Z>x\} = P\{Z<-x\}$

In pictures

$P\{Z>x\}=$ 

$=$ 

$= P\{Z<-x\}$

With integrals

$$P\{Z>x\} = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-y^2/2}\,dy$$

$u=-y$
$du=-dy$

$$= \frac{1}{\sqrt{2\pi}} \int_{-x}^{-\infty} e^{-u^2/2}(-du)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} e^{-u^2/2}\,du = P\{Z<-x\}$$

(b) $P\{|Z|>x\} = 2P\{Z>x\}$

Proof $|Z|>x$ iff $Z>x$ or $-Z>x$

iff $Z>x$ or $Z<-x$

$$P\{|Z|>x\} = P\{Z>x\}+P\{Z<-x\}$$

$$= P\{Z>x\}+P\{Z>x\} \quad \text{by part (a)}$$

$$= 2P\{Z>x\} \quad \text{QED}$$

(c) $P\{|Z|<x\} = 2P\{Z<x\}-1$

Proof: $P\{|Z|<x\} = 1-P\{|Z|>x\}$

$$= 1-2P\{Z>x\} \quad \text{by part (b)}$$

$$= 1-2\left[1-P\{Z<x\}\right]$$

$$= 1-2+2P\{Z<x\}$$

$$= 2P\{Z<x\}-1 \quad \text{QED}$$

Solutions 11

Problems

5.32 X = time in hours to repair machine is exponential with $\lambda = \frac{1}{2}$

(a) $P(X > 2) = 1 - F(2) = 1 - \left[1 - e^{-\lambda \cdot 2}\right] = e^{-\lambda \cdot 2} = e^{-\frac{1}{2} \cdot 2} = e^{-1}$

(b) $P(X > 10 \mid X > 9) = P(X > 1)$ by memorylessness

$= e^{-\lambda \cdot 1} = e^{-(\frac{1}{2})}$

5.34 X = # miles (in thousands) car can be driven before complete breakdown.

Assume X is exponential with $\lambda = \frac{1}{20}$

Assume car has 10,000 miles already, what is probability it last another 20,000?

$P(X > 30 \mid X > 10) = P(X > 20)$ by memorylessness

$= e^{-\lambda(20)} = e^{-\frac{1}{20} 20} = e^{-1}$

Now assume X is uniformly distributed in the interval $(0, 40)$

$$P(X > 30 \mid X > 10) = \frac{P(X > 30)}{P(X > 10)} = \frac{(10/40)}{(30/40)} = \frac{10}{30} = \frac{1}{3}$$

since: $P(X > 30) = \frac{40-30}{40-0} = \frac{10}{40} = \frac{1}{4}$ $\quad P(X > 10) = \frac{40-10}{40-0} = \frac{30}{40} = \frac{3}{4}$

5.39 X is exponential with parameter $\lambda = 1$

$$f_X = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \le 0 \end{cases} = \begin{cases} e^{-x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

$Y = \log X$. find density function of Y

here $g(x) = \log x$ $\qquad g^{-1}(y) = e^y$

$$(g^{-1})'(y) = e^y$$

So $f_Y(y) = f_X(g^{-1}(y)) \cdot (g^{-1})'(y) = f_X(e^y) \cdot e^y$

$$= e^{-e^y} \cdot e^y = e^{y - e^y}$$

Theoretical exercises

5.13 Median is m such that $F(m) = \frac{1}{2}$

(a) X is uniform over (a, b)

$$F_X = \begin{cases} 0 & x \le a \\ \frac{x-a}{b-a} & a < x \le b \\ 1 & b < x \end{cases}$$

so $F(m) = \frac{1}{2} \Rightarrow \frac{m-a}{b-a} = \frac{1}{2} \Rightarrow m - a = \frac{1}{2}(b-a) \Rightarrow m = \frac{1}{2}(b+a)$

$\therefore$ m = median = midpoint of (a, b)

(b) X normal with mean μ and variance σ^2

$$F_X(x) = P\{X \le x\} = P\{Z \le \frac{x-\mu}{\sigma}\} = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

standard normal

$$\frac{1}{2} = F_X(m) = \Phi\left(\frac{m-\mu}{\sigma}\right) \Rightarrow \frac{m-\mu}{\sigma} = \Phi^{-1}\left(\frac{1}{2}\right)$$

Now $\Phi(0) = \frac{1}{2}$, so $\Phi^{-1}\left(\frac{1}{2}\right) = 0$

$\Rightarrow \frac{m-\mu}{\sigma} = 0 \Rightarrow m = \mu$ so median = mean for normal random variable

(c) X is exponential with rate λ.

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

$$\frac{1}{2} = F_X(m) = 1 - e^{-\lambda m} \Rightarrow e^{-\lambda m} = \frac{1}{2} \Rightarrow -\lambda m = \log \frac{1}{2} = -\log 2$$

$\Rightarrow m = \frac{\log 2}{\lambda}$. Note: Expectation $= \frac{1}{\lambda}$, so median $\neq$ mean for exponential dist.

5.30 X has density function f_X. Let $Y = aX + b$. Find f_Y, the density function of Y

Note: I will assume $a > 0$, for in this case $g(x) = ax + b$ is an increasing function, which was the only situation we discussed in class. If $a < 0$ there is a similar solution, while if $a=0$ then Y does not have a density function, as it is constant, and hence discarde.

Assume $a>0$, then $g(x)=ax+b$ is an increasing functi

$$g^{-1}(y) = \frac{x-b}{a} \qquad (g^{-1})'(y) = \frac{1}{a}$$

$$f_Y(y) = f_X(g^{-1}(y))\,(g^{-1})'(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

If $a<0$, then $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

If $a=0$, then $Y=b$ is a discrete R.V. with probability mass functi $p(y) = \begin{cases} 1 & y=b \\ 0 & y \neq b \end{cases}$

5.31 X normal R.V. with mean μ and variance σ^2

$Y = e^X \qquad g(x) = e^x \qquad$ range $= (0,\infty)$

$$g^{-1}(y) = \log y \qquad (g^{-1})'(y) = \frac{1}{y}$$

So for $y \in (0,\infty)$

$$f_X = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

$$f_Y(y) = f_X(\log y)\,\frac{1}{y}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \frac{1}{y} \cdot \exp\left\{-\frac{(\log y - \mu)^2}{2\sigma^2}\right\} \quad \text{for } y>0$$

$$f_Y(y) = 0 \quad \text{for } y \leq 0$$

Ch 6 problems

6.1 2 fair dice are rolled: find joint probability mass function if:

(a) X = max value, Y = sum of values

X \ Y	2	3	4	5	6	7	8	9	10	11	12
1	1/36	0	0	0	0	0	0	0	0	0	0
2	0	2/36	1/36	0	0	0	0	0	0	0	0
3	0	0	2/36	2/36	1/36	0	0	0	0	0	0
4	0	0	0	2/36	2/36	2/36	1/36	0	0	0	0
5	0	0	0	0	2/36	2/36	2/36	2/36	1/36	0	0
6	0	0	0	0	0	2/36	2/36	2/36	2/36	2/36	1/36

(b) X = first die, Y = max value

X \ Y	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	0	2/36	1/36	1/36	1/36	1/36
3	0	0	3/36	1/36	1/36	1/36
4	0	0	0	4/36	1/36	1/36
5	0	0	0	0	5/36	1/36
6	0	0	0	0	0	6/36

(c) X = min value
Y = max value

X \ Y	1	2	3	4	5	6
1	1/36	2/36	2/36	2/36	2/36	2/36
2	0	1/36	2/36	2/36	2/36	2/36
3	0	0	1/36	2/36	2/36	2/36
4	0	0	0	1/36	2/36	2/36
5	0	0	0	0	1/36	2/36
6	0	0	0	0	0	1/36

6.7 X_1 = # failures before first success
X_2 = # failures between first and second successes.

X_1 and X_2 are discrete RVs with possible values
$n, m = 0, 1, 2, \ldots$

Joint PMF: $p(n,m) = P\{X_1 = n, X_2 = m\}$

$$= P(\underbrace{FF\cdots F}_{n} S \underbrace{FF\cdots F}_{m} S) = (1-p)^n p (1-p)^m p$$

So $p(n,m) = (1-p)^{n+m} p^2$

6.9 Joint density is

$$f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \quad \text{for } 0<x<1,\ 0<y<2$$

(a) Verify normalization:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^2\int_0^1 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dx\,dy$$

$$= \frac{6}{7}\int_0^2 \left[\frac{x^3}{3} + \frac{x^2 y}{4}\right]_{x=0}^{x=1} dy = \frac{6}{7}\int_0^2 \left[\frac{1}{3} + \frac{y}{4}\right] dy$$

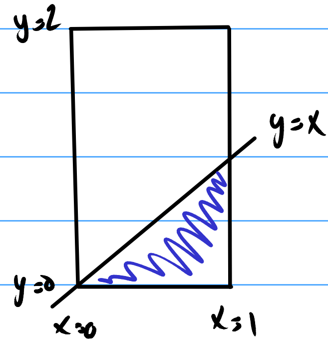
$$= \frac{6}{7}\left[\frac{y}{3} + \frac{y^2}{8}\right]_{y=0}^{y=2} = \frac{6}{7}\left[\frac{2}{3} + \frac{4}{8}\right] = \frac{6}{7}\left[\frac{2}{3} + \frac{1}{2}\right] = \frac{6}{7}\left[\frac{7}{6}\right] = 1. \checkmark$$

(b) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^2 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy$

$$= \frac{6}{7}\left[x^2 y + \frac{xy^2}{4}\right]_{y=0}^{y=2} = \frac{6}{7}\left[2x^2 + x\right]$$

(c) $P\{X > Y\} = \iint_{\{x>y\}} f(x,y)\,dx\,dy = \iint_{\substack{0<x<1\\ 0<y<2\\ x<y}} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dx\,dy$

Picture of region



$$= \int_0^1 \int_0^x \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy\,dx$$

$$= \frac{6}{7}\int_0^1 \left[x^2 y + \frac{xy^2}{4}\right]_{y=0}^{y=x} dx = \frac{6}{7}\int_0^1 \left[x^3 + \frac{x^3}{4}\right] dx$$

$$= \frac{6}{7}\cdot\frac{5}{4}\cdot\int_0^1 x^3\,dx = \frac{6}{7}\cdot\frac{5}{4}\cdot\left[\frac{x^4}{4}\right]_{x=0}^{x=1} = \frac{\cancel{6}^{3}}{7}\cdot\frac{5}{\cancel{4}_{2}}\cdot\frac{1}{4} = \frac{15}{56}$$

(d) $P\left(Y > \frac{1}{2} \mid X < \frac{1}{2}\right) = \dfrac{P\left(Y > \frac{1}{2} \text{ and } X < \frac{1}{2}\right)}{P\left(X < \frac{1}{2}\right)}$

$$P\left(X < \frac{1}{2}\right) = \int_0^{1/2} f_X(x)\,dx = \int_0^{1/2} \frac{6}{7}\left[2x^2 + x\right] dx$$

$$= \frac{6}{7}\left[\frac{2}{3}x^3 + \frac{x^2}{2}\right]_{x=0}^{x=1/2} = \frac{6}{7}\left[\frac{2}{3}\cdot\frac{1}{8} + \frac{1}{8}\right] = \frac{6}{7}\cdot\frac{5}{3}\cdot\frac{1}{\cancel{8}_{4}} = \frac{5}{28}$$

$$P\left(Y>\tfrac{1}{2}, X<\tfrac{1}{2}\right) = \int_{1/2}^{2}\int_{0}^{1/2} \frac{6}{7}\left(x^2+\frac{xy}{2}\right)dx\,dy$$

$$= \frac{6}{7}\int_{1/2}^{2}\left[\frac{x^3}{3}+\frac{x^2y}{4}\right]_{x=0}^{x=1/2} dy = \frac{6}{7}\int_{1/2}^{2}\left[\frac{1}{24}+\frac{y}{16}\right]dy$$

$$= \frac{6}{7}\left[\frac{y}{24}+\frac{y^2}{32}\right]_{1/2}^{2} = \frac{6}{7}\left[\frac{1}{12}+\frac{1}{8}-\frac{1}{48}-\frac{1}{128}\right] = \frac{69}{448}$$

$$P\left(Y>\tfrac{1}{2} \mid X<\tfrac{1}{2}\right) = \frac{69/448}{5/28} = \frac{69}{80}$$

(e) $$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x\cdot\frac{6}{7}\cdot(2x^2+x)\,dx$$

$$= \int_0^1 \frac{6}{7}(2x^3+x^2)\,dx = \frac{6}{7}\left[\frac{x^4}{2}+\frac{x^3}{3}\right]_{x=0}^{x=1} = \frac{6}{7}\left[\frac{1}{2}+\frac{1}{3}\right] = \frac{5}{7}$$

(f) $$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \frac{6}{7}\left(x^2+x\frac{y}{2}\right)dx$$

$$= \frac{6}{7}\left[\frac{x^3}{3}+\frac{x^2y}{4}\right]_{x=0}^{x=1} = \frac{6}{7}\left[\frac{1}{3}+\frac{y}{4}\right]$$

$$E[Y] = \int_0^2 y\cdot\frac{6}{7}\cdot\left[\frac{1}{3}+\frac{y}{4}\right] = \frac{6}{7}\int_0^2\left(\frac{y}{3}+\frac{y^2}{4}\right)dy$$

$$= \frac{6}{7}\left[\frac{y^2}{6} + \frac{y^3}{12}\right]_0^2 = \frac{6}{7}\left[\frac{4}{6} + \frac{8}{12}\right] = \frac{8}{7}$$

Solutions 12

Problems

6.18 Two points on Line : X uniform over $(0, \frac{L}{2})$
Y uniform over $(\frac{L}{2}, L)$

X and Y are independent

Distance between $= Y - X$ (since Y is to right of X)

$P\{Y - X > \frac{L}{3}\}$?

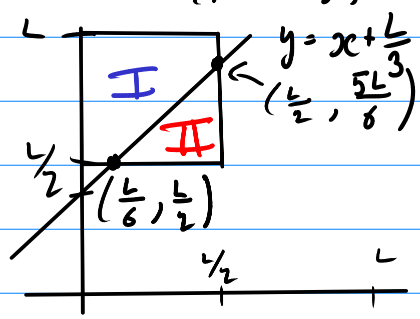
Density function : $f_X(x) = \begin{cases} 2/L & 0 < x < L/2 \\ 0 & \text{otherwise} \end{cases}$

$$f_Y(y) = \begin{cases} 2/L & L/2 < y < L \\ 0 & \text{otherwise} \end{cases}$$

Joint density : $f(x,y) = \begin{cases} \frac{4}{L^2} & 0 < x < \frac{L}{2} \text{ and } \frac{L}{2} < y < L \\ 0 & \text{otherwise} \end{cases}$

$$P\{Y - X > \tfrac{L}{3}\} = \iint_{\{y - x > \frac{L}{3}\}} f(x,y)\,dx\,dy = \iint_{\substack{y - x > \frac{L}{3} \\ 0 < x < \frac{L}{2} \\ \frac{L}{2} < y < L}} \frac{4}{L^2}\,dx\,dy$$

Region



$= \frac{4}{L^2}\text{Area}(\text{I})$

$= 1 - \frac{4}{L^2}\text{Area}(\text{II})$

$$\text{Area}(\text{II}) = \frac{1}{2}\left(\frac{L}{3}\right)^2 = \frac{1}{2}\frac{L^2}{9}$$

$P\{Y-X > \frac{L}{3}\} = 1 - \frac{4}{L^2}\left(\frac{1}{2}\,\frac{L^2}{9}\right) = 1 - \frac{2}{9} = \frac{7}{9}$

6.20 joint density $f(x,y) = \begin{cases} xe^{-(x+y)} & x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$

X and Y are independent

$x>0$: $f_X(x) = \int_0^\infty xe^{-(x+y)}\,dy = xe^{-x}\int_0^\infty e^{-y}\,dy = xe^{-x}[0+1] = xe^{-x}$

$x<0$ $f_X(x) = 0$

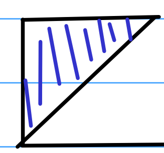
$y>0$ $f_Y(y) = \int_0^\infty xe^{-(x+y)}\,dx = e^{-y}\int_0^\infty xe^{-x}\,dx = e^{-y}\left[-xe^{-x} - e^{-x}\right]_{x=0}^{\infty}$

$= e^{-y}[-0-0+0+1] = e^{-y}$

$y<0$ $f_Y(y) = 0.$

So $f(x,y) = f_X(x)\cdot f_Y(y)$, and the variables are independent

In the case $f(x,y) = \begin{cases} 2 & 0<x<y,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$



Well: $f_X(x) = \begin{cases} \int_x^1 2\,dy & \text{if } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

$f_X(x) = \begin{cases} 2(1-x) & \text{if } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

$$f_Y(y) = \begin{cases} \int_0^y 2\,dx & \text{if } 0<y<1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2y & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore f_X(x)f_Y(y) = \begin{cases} 4(1-x)y & \text{if } 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

and $f_X(x)f_Y(y) \neq f(x,y)$

So X and Y are not independent.

6.29 X = weekly sales is normal w/ $\mu = 2200$ $\sigma = 230$

(a) $P\{$total sales over two weeks $> 5000\}$

let X_1 = sales in first week, X_2 = sales in second week $\}$ each normal $\mu = 2200$ $\sigma = 230$

Assuming X_1 and X_2 are independent:

$X_1 + X_2$ is normal with $\mu = 2200 + 2200 = 4400$

$$\sigma = \sqrt{(230)^2 + (230)^2} \approx 325.3$$

$$P\{X_1 + X_2 > 5000\} = P\left\{Z > \frac{5000-4400}{325.3}\right\} \approx P\{Z > 1.84\}$$

$$= 1 - P\{Z < 1.84\} = 1 - \Phi(1.84) \approx 1 - .97 = .03$$

(b) $P\{$sales > 2000 in atleast 2 out of 3 weeks$\}$

X_1, X_2, X_3 sales in the 3 weeks.

For each week: $P\{X_i > 2000\} = P\{Z > \frac{2000-2200}{230}\} \approx P\{Z > -.87\}$

$= P\{Z < .87\} = \Phi(.87) \approx .808$

$P\{$at least 2 out of 3 are $> 2000\}$

$= P\{X_1 > 2000, X_2 > 2000, X_3 \leq 2000\}$

$+ P\{X_1 > 2000, X_2 \leq 2000, X_3 > 2000\}$

$+ P\{X_1 \leq 2000, X_2 > 2000, X_3 > 2000\}$

$+ P\{X_1 > 2000, X_2 > 2000, X_3 > 2000\}$

$= 3(.808)^2(1-.808) + (.808)^3$ { assuming independence.

$= 3 \cdot (.653) \cdot (.192) + (.528) = .904$

6.30 X = Jill's score is normal w/ $\mu = 170$ $\sigma = 20$
Y = Jack's score is normal w/ $\mu = 160$ $\sigma = 15$

Assume X and Y independent.

$P\{$Jack's score is higher$\} = P\{Y > X\} = P\{X - Y < 0\}$

Y normal w/ $\mu = 160$ $\sigma = 15 \Rightarrow -Y$ normal w/ $\mu = -160$, $\sigma = 15$

$\Rightarrow X - Y = X + (-Y)$ is normal w/ $\mu = 170 + (-160) = 10$

$\sigma = \sqrt{20^2 + 15^2} = 25$

$$P\{X-Y<0\} \underbrace{=}_{\text{continuity correction}} P\{X-Y<-.5\} = P\{Z<\frac{-.5-10}{25}\} = P\{Z<-.42\}$$

$$= P\{Z>.42\} = 1-P\{Z<.42\} = 1-\Phi(.42) = 1-.6628$$

$$= .3372$$

(b) $P\{X+Y>350\}$?

$X+Y$ is normal w/ $\mu = 170+160 = 330$ $\sigma = \sqrt{20^2+15^2} = 25$

continuity correction

so $P\{X+Y>350\} = P\{X+Y>350.5\} = P\{Z>\frac{350.5-330}{25}\} = P\{Z>.82\}$

$$= 1-\Phi(.82) \approx 1-.7939 = .2061$$

6.32 X = # typos on page has expected value .2

** We will model X as a Poisson RV. with $\lambda = .2$

Article has 10 pages : X_i = # typos on ith page.

** Assume pages are independent ; so all X_i are independent

Sum of Poisson RV-s is poisson

so X_1+X_2 Poisson with $\lambda = .2+.2 = .4$

$\underbrace{X_1+X_2+\cdots+X_9+X_{10}}_{10}$ is Poisson with $\lambda = \underbrace{.2+\cdots+.2}_{10} = 2$

$$P\left\{\sum_{i=1}^{10} X_i = 0\right\} = e^{-2}$$

$$P\left\{\sum_{i=1}^{10} X_i \geq 2\right\} = 1 - P\left\{\sum_{i=1}^{10} X_i = 0\right\} - P\left\{\sum_{i=1}^{10} X_i = 1\right\}$$

$$= 1 - e^{-2} - e^{-2}\frac{2^1}{1!} = 1 - 3e^{-2}$$

6.38 Choose $X \in \{1,2,3,4,5\}$ then choose $Y \leq X$

(a) joint PMF

x \ y	1	2	3	4	5	Px
1	1/5	0	0	0	0	1/5
2	1/10	1/10	0	0	0	1/5
3	1/15	1/15	1/15	0	0	1/5
4	1/20	1/20	1/20	1/20	0	1/5
5	1/25	1/25	1/25	1/25	1/25	1/5
Py	.457	.257	.157	.09	.04	

probability may not add to 1 due to rounding

(b) Conditional mass function of X

$p(x|y)$

x \ y	1	2	3	4	5
1	.438	0	0	0	0
2	.219	.389	0	0	0
3	.146	.259	.425	0	0
4	.109	.195	.318	.556	0
5	.088	.156	.255	.444	1

(c) Are X and Y independent: NO, $p(x|y)$ depends very much on the value of Y.

6.42 Joint density is given by

$$f(x,y) = c(x^2-y^2)e^{-x} \quad \text{for } 0 \le x < \infty \quad -x \le y \le x$$

$$f_X(x) = \int_{-x}^{x} c(x^2-y^2)e^{-x}\,dy = ce^{-x}\int_{-x}^{x}(x^2-y^2)\,dy$$

$$= ce^{-x}\left[x^2y - \frac{y^3}{3}\right]_{y=-x}^{y=x} = ce^{-x}\left[x^3 - \frac{x^3}{3} - x^2(-x) + \frac{(-x)^3}{3}\right]$$

$$= ce^{-x}\left[x^3 - \frac{x^3}{3} + x^3 - \frac{x^3}{3}\right] = ce^{-x}\cdot\frac{4}{3}x^3$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{c(x^2-y^2)e^{-x}}{ce^{-x}\cdot\frac{4}{3}\cdot x^3} = \frac{3}{4}\left(\frac{1}{x} - \frac{y^2}{x^3}\right)$$

This is for $-x \le y \le x$ $\quad f(y|x) = 0$ otherwise

Distribution $F_{Y|X}(a|x) = \int_{-\infty}^{a} f(y|x)\,dy$

$$= \int_{-x}^{a} \frac{3}{4}\left(\frac{1}{x} - \frac{y^2}{x^3}\right)dy = \frac{3}{4}\left[\frac{y}{x} - \frac{y^3}{3x^3}\right]_{y=-x}^{y=a}$$

$$= \frac{3}{4}\left[\frac{a}{x} - \frac{a^3}{3x^3} - \frac{-x}{x} + \frac{(-x)^3}{3x^3}\right] = \frac{3}{4}\left[\frac{a}{x} - \frac{a^3}{3x^3} + 1 - \frac{1}{3}\right]$$

$$F(y|x) = \frac{3a}{4x} - \frac{a^3}{4x^3} + \frac{1}{2} \quad (\text{valid for } -x \le a \le x)$$

$$F(y|x) = 0 \text{ if } a < -x$$

$$F(y|x) = 1 \text{ if } a > x$$

Theoretical exercises

6.11 X_1, X_2, X_3, X_4, X_5 be independent continuous R.V.s having common distribution F and density f

$$I = P\{X_1 < X_2 < X_3 < X_4 < X_5\}$$

(a) $$I = \underset{\{x_1<x_2<x_3<x_4<x_5\}}{\iiiint\!\!\int} f(x_1)f(x_2)f(x_3)f(x_4)f(x_5)\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{x_5}\int_{-\infty}^{x_4}\int_{-\infty}^{x_3}\int_{-\infty}^{x_2} f(x_1)\cdots f(x_5)\,dx_1\,dx_2\,dx_3\,dx_4\,dx_5$$

substitute:

$u_i = F(x_i) \qquad du_i = f(x_i)\,dx_i$

$x_i = -\infty \iff u_i = 0 \qquad x_5 = \infty \iff u_5 = 1$

$x_i < x_{i+1} \iff u_i < u_{i+1}$

$$I = \int_0^1\int_0^{u_5}\int_0^{u_4}\int_0^{u_3}\int_0^{u_2} du_1\,du_2\,du_3\,du_4\,du_5$$

so I does not depend on F or f!

(b) $\int_0^{u_2} du_1 = u_2$, $\int_0^{u_3} u_2\, du_2 = \frac{u_3^2}{2}$, $\int_0^{u_4} \frac{u_3^2}{2} du_3 = \frac{u_4^3}{6}$

$\int_0^{u_5} \frac{u_4^3}{6} du_4 = \frac{u_5^4}{24}$, $\int_0^1 \frac{u_5^4}{24} = \frac{1}{120} = \frac{1}{5!}$

(c) Intuitive explanation: since $X_1, \dots, X_5$ all have same distribution, each of the $5!$ numerical orders of the values is equally likely to occur.

since $X_1 < X_2 < X_3 < X_4 < X_5$ is one of these orders, it's probability is $\frac{1}{5!}$

6.14 X and Y independent geometric random variables with same parameter p.

(a) $P\{X=i \mid X+Y=n\}$? In Bernoulli trials w/ parameter p, X is # of trials to first success, and X+Y is number of trials to second success. So given that second success occurs on n th trial, there are (n-1) possible trials where the first success could have occurred. It seems that any of these (n-1) possibilities is equally likely, so we conjecture

$$P\{X=i \mid X+Y=n\} = \frac{1}{n-1}$$

(b) <u>Proof</u>: $P\{X=i \mid X+Y=n\} = \dfrac{P\{X=i, X+Y=n\}}{P\{X+Y=n\}} = \dfrac{P\{X=i, Y=n-i\}}{P\{X+Y=n\}}$

$= \dfrac{P\{X=i\}P\{n-i\}}{P\{X+Y=n\}}$ since X and Y are independent.

$P\{X=i\} = (1-p)^{i-1}p \qquad P\{Y=n-i\} = (1-p)^{n-i-1}p$

$P\{X+Y=n\} = \sum_{i=1}^{n-1} P\{X=i\}P\{Y=n-i\} = \sum_{i=1}^{n-1} (1-p)^{i-1}p(1-p)^{n-i-1}p$

$= \sum_{i=1}^{n-1} (1-p)^{n-2}p^2 = (n-1)(1-p)^{n-2}p^2$

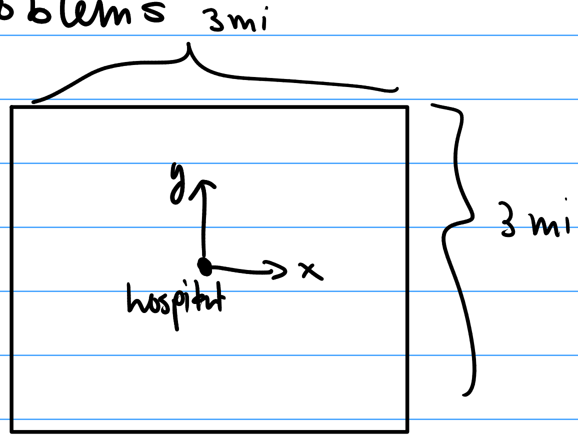
[Could also say $X+Y$ is negative binomial w/ p, $r=2$]

$P\{X=i \mid X+Y=n\} = \dfrac{(1-p)^{i-1}p\,(1-p)^{n-i-1}p}{(n-1)(1-p)^{n-2}p^2} = \dfrac{1}{n-1}$ ✓

Solutions 13

Problems

7.5



- Hospital at $(0,0)$
- Accident at (X,Y) uniformly in the square
  $-1.5 < X < 1.5$
  $-1.5 < Y < 1.5$
- Due to rectangular street grid, travel distance is $|X| + |Y|$

Expected distance

$$E\big[|X| + |Y|\big] = \iint (|x| + |y|)\, f(x,y)\, dx\, dy$$

Now, $f(x,y) = \begin{cases} \frac{1}{9} & \text{if } \begin{array}{l} -1.5 < x < 1.5 \\ -1.5 < y < 1.5 \end{array} \\ 0 & \text{otherwise} \end{cases}$

so $E\big[|X| + |Y|\big] = \int_{-1.5}^{1.5} \int_{-1.5}^{1.5} (|x| + |y|) \frac{1}{9}\, dx\, dy$

Break into 4 parts depending on signs of x and y:

$x > 0, y > 0$: $\int_0^{1.5} \int_0^{1.5} (x+y) \frac{1}{9}\, dx\, dy = \frac{1}{9} \int_0^{1.5} \left[\frac{x^2}{2} + xy \right]_0^{1.5} dy$

$= \frac{1}{9} \int_0^{1.5} \left[\frac{(1.5)^2}{2} + 1.5\, y \right) dy = \frac{1}{9} \left[\frac{1.5^2}{2} y + 1.5 \frac{y^2}{2} \right]_0^{1.5} = \frac{1}{9} \left[\frac{(1.5)^3}{2} + \frac{(1.5)^3}{2} \right]$

$= \frac{1}{9} (1.5)^3 = \frac{1}{9} \cdot \frac{3^3}{2^3} = \frac{3}{8}$

By symmetry, the other 3 regions all give the same value $\frac{3}{8}$

- $x>0, y<0$
- $x<0, y>0$
- $x<0, y<0$

so $E\left[|X|+|Y|\right] = 4 \cdot \frac{3}{8} = \boxed{\frac{3}{2}}$

7.9 Balls $1, \dots, n$ placed in urns $1, \dots, n$ in such a way that ball i has equal chance of being in urns $1, \dots, i$.

(a) Expected # of urns that are empty $= X$

$X = X_1 + X_2 + \cdots + X_n$

where $X_i = \begin{cases} 1 & \text{if urn } i \text{ is empty} \\ 0 & \text{otherwise} \end{cases}$

$E[X_i] = 0 P\{X_i = 0\} + 1 \cdot P\{X_i = 1\} = P\{X_i = 1\}$

Need $P\{X_i = 1\} = P\{\text{urn } i \text{ is empty}\}$

$= P\{\text{ball } i \text{ not in urn } i\} P\{\text{ball } i+1 \text{ not in urn } i\} \dots P\{\text{ball } n \text{ not in urn } i\}$

$= \frac{i-1}{i} \cdot \frac{i}{i+1} \cdot \frac{i+1}{i+2} \cdot \dots \cdot \frac{n-1}{n} = \frac{i-1}{n}$

So $E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{(i-1)}{n} = \frac{1}{n}\left[\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right] = \frac{1}{n}\left[\frac{(n+1)n}{2} - n\right]$

$= \frac{n+1}{2} - 1 = \boxed{\frac{n-1}{2}}$

(b) $P\{$none of the urns are empty$\}$

No urn empty $\Rightarrow$ ball n in urn n $\Rightarrow$ ball $n-1$ in urn $n-1$ $\Rightarrow$ ball $n-2$ in urn $n-2$ $\Rightarrow \cdots \Rightarrow$ ball 1 in urn 1

so Probability $= \frac{1}{n} \cdot \frac{1}{n-1} \cdot \frac{1}{n-2} \cdots \frac{1}{1} = \boxed{\frac{1}{n!}}$

7.11 n independent flips of a coin ($P(\text{heads}) = p$)

$X = \#$ changeovers ($\cdots HT \cdots$ or $\cdots TH \cdots$)

Let $X_i = \begin{cases} 1 & \text{if change over between } i \text{ and } i+1 \\ 0 & \text{otherwise} \end{cases}$

$E[X_i] = P\{X_i = 1\} = P(HT \text{ or } TH) = p(1-p) + (1-p)p = 2p(1-p)$

For n flips we have $(n-1)$ such variables, as there are $n-1$ places where a changeover can happen

$X = \sum_{i=1}^{n-1} X_i$

$E[X] = \sum_{i=1}^{n-1} E[X_i] = \sum_{i=1}^{n-1} 2p(1-p) = \boxed{2(n-1)p(1-p)}$

Theoretical Exercises

7.4 X has finite mean μ and variance σ^2.

$g(x) = g(\mu) + g'(\mu)(x-\mu) + \frac{g''(\mu)}{2}(x-\mu)^2 + \cdots$

$$E[g(X)] = E\left[g(\mu) + g'(\mu)(X-\mu) + \frac{g''(\mu)}{2}(X-\mu)^2 + \cdots\right]$$

$$= g(\mu) + g'(\mu)\,E[X-\mu] + \frac{g''(\mu)}{2}\,E[(X-\mu)^2] + \cdots$$

$$= g(\mu) + \frac{g''(\mu)}{2}\,\sigma^2$$

since $E[X-\mu] = E[X] - \mu = \mu - \mu = 0$

and $E[(X-\mu)^2] = 0$ by definition.

7.5 $A_1, \ldots, A_n$ events

C_k = event that at least k A_i's occur.

Let $X = \#\, A_i$'s that occur $\quad X_i = \begin{cases} 1 & \text{if } A_i \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$

Then $X = \sum_{i=1}^{n} X_i$

$$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} P\{X_i = 1\} = \sum_{i=1}^{n} P(A_i)$$

On the other hand: $C_k = P\{X \geq k\}$

$$\sum_{k=1}^{n} P(C_k) = \sum_{k=1}^{n} P\{X \geq k\} = \sum_{k=1}^{n} \sum_{j=k}^{n} P\{X = j\}$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{j} P\{X = j\}$$

sum over pairs $k \leq j$
reverse order of summation

$$= \sum_{j=1}^{n} j \cdot P\{X = j\} = E[X] \text{ by definition}$$

Hence $\sum_{k=1}^{n} P(C_k) = E[X] = \sum_{i=1}^{n} P(A_i)$

Solutions 14

7.32 $X_i = \begin{cases} 1 & \text{if } i\text{th urn is empty} \\ 0 & \text{otherwise} \end{cases}$

$E[X_i] = P\{i\text{th urn is empty}\} = \frac{i-1}{n}$ from previous HW set.

$X = \sum_{i=1}^{n} X_i = \#$ of empty urns

$Var(X) = Cov(X,X) = Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j\right)$

$= \sum_{i=1}^{n} Cov(X_i, X_i) + 2\sum_{i<j} Cov(X_i, X_j)$

$= \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)$

$Var(X_i) = E[X_i^2] - E[X_i]^2$

$X_i^2 = X_i$ since X_i has values 0 or 1,

so $E[X_i^2] = \frac{i-1}{n}$

$Var(X_i) = \frac{i-1}{n} - \left(\frac{i-1}{n}\right)^2 = \frac{i-1}{n}\left(1 - \frac{i-1}{n}\right)$

$Cov(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j]$

$X_i X_j = \begin{cases} 1 & \text{if both urn } i \text{ and urn } j \text{ are empty} \\ 0 & \text{otherwise} \end{cases}$

$E[X_i X_j] = P\{\text{both urn } i \text{ and urn } j \text{ are empty}\}$

Assume $i < j$. Then all balls before ith cannot go into either, all balls from i to $j-1$ can go into i but not j, balls from j to n can go into either

$$P\{i \text{ and } j \text{ empty}\} = \underbrace{\frac{i-1}{\cancel{i}} \frac{\cancel{i}}{\cancel{i+1}} \cdots \frac{\cancel{j-2}}{j-1}}_{\text{simplifies to } \frac{i-1}{j-1}} \underbrace{\frac{j-2}{\cancel{j}} \frac{j-1}{j+1} \frac{\cancel{j}}{j+2} \cdots \frac{\cancel{n-3}}{n-1} \frac{\cancel{n-2}}{n}}_{\text{simplifies to } \frac{(j-2)(j-1)}{(n-1)n}}$$

$$= \frac{(i-1)(j-2)}{(n-1)\,n}$$

$$E[X_i] = \frac{i-1}{n} \qquad E[X_j] = \frac{j-1}{n}$$

$$\text{Cov}(X_i, X_j) = \frac{(i-1)(j-2)}{(n-1)\,n} - \frac{i-1}{n}\,\frac{j-1}{n}$$

$$= \frac{i-1}{n}\left(\frac{j-2}{n-1} - \frac{j-1}{n}\right)$$

So $\text{Var}(X) = \sum_{i=1}^{n} \frac{i-1}{n}\left(1 - \frac{i-1}{n}\right) + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{i-1}{n}\left(\frac{j-2}{n-1} - \frac{j-1}{n}\right)$

7.36 X = # of 1's in n rolls of fair die
Y = # of 2's in n rolls of fair die

Let $X_i = \begin{cases} 1 & \text{if } i\text{th roll is } 1 \\ 0 & \text{otherwise} \end{cases}$ $\quad Y_j = \begin{cases} 1 & \text{if } j\text{th roll is } 2 \\ 0 & \text{otherwise} \end{cases}$

So $X = \sum_{i=1}^{n} X_i \qquad Y = \sum_{j=1}^{n} Y_j$

$$Cov(X,Y) = Cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} Y_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} Cov(X_i, Y_j)$$

if $i \neq j$, then X_i and Y_j are independent, so $Cov(X_i, Y_j) = 0$.

if $i = j$, then X_i and Y_i are actually dependent

$$Cov(X_i, Y_i) = E[X_i Y_i] - E[X_i]E[Y_i]$$

Now $E[X_i] = P\{i\text{th roll is } 1\} = 1/6$
$E[Y_i] = P\{i\text{th roll is } 2\} = 1/6$

Also $X_i Y_i = \begin{cases} 1 & \text{if } i\text{th roll is both a 1 and a 2} \\ 0 & \text{otherwise} \end{cases}$

So $X_i Y_i = 1$ is actually impossible!
$P\{X_i Y_i = 1\} = 0$, so $E[X_i Y_i] = 0$

Thus $Cov(X_i, Y_i) = 0 - \frac{1}{6}\frac{1}{6} = -\frac{1}{36}$

So $Cov(X,Y) = \sum_{i=1}^{n} Cov(X_i, Y_i) = n\left(-\frac{1}{36}\right) = \frac{-n}{36}$

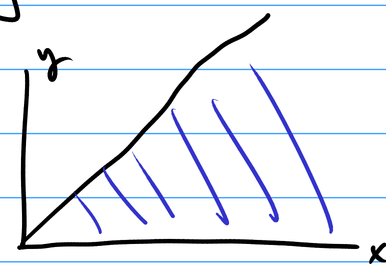
7.38 X and Y have joint density

$$f(x,y) = \begin{cases} 2e^{-2x}/x & 0 \le x < \infty,\ 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$E[X] = \iint x f(x,y)\,dx\,dy$$

Region



$$= \int_0^\infty \int_0^x x \cdot \frac{2e^{-2x}}{x}\,dy\,dx$$

$$= \int_0^\infty \int_y^\infty 2e^{-2x}\,dx\,dy = \int_0^\infty \left[-e^{-2x}\right]_y^\infty dy$$

$$= \int_0^\infty e^{-2y}\,dy = \left[-\tfrac{1}{2}e^{-2y}\right]_0^\infty = 0 - \left(-\tfrac{1}{2}\right)e^{-2\cdot 0} = \frac{1}{2}$$

$$E[Y] = \iint y f(x,y)\,dx\,dy = \int_0^\infty \int_0^x y\,\frac{2e^{-2x}}{x}\,dy\,dx$$

$$= \int_0^\infty \frac{2e^{-2x}}{x}\left[\frac{y^2}{2}\right]_0^x dx = \int_0^\infty \frac{x^2}{2}\,\frac{2e^{-2x}}{x}\,dx$$

$$= \int_0^\infty x e^{-2x}\,dx = \left[-\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}\right]_0^\infty$$

$$= 0 - 0 - \left(-0 - \frac{1}{4}\right) = \frac{1}{4}$$

$$E[XY] = \iint xy\, f(x,y)\,dx\,dy$$

$$= \int_0^\infty \int_0^x xy\, \frac{2e^{-2x}}{x}\, dy\, dx = \int_0^\infty \int_0^x y \cdot 2e^{-2x}\, dy\, dx$$

$$= \int_0^\infty \left[2e^{-2x} \frac{y^2}{2}\right]_{y=0}^{y=x} dx = \int_0^\infty x^2 e^{-2x}\, dx$$

$$= \left[-\frac{1}{4} e^{-2x}\left(1+2x+2x^2\right)\right]_0^\infty = 0 + \frac{1}{4}e^{-2\cdot 0}\left(1+2\cdot 0+2\cdot 0^2\right)\Big]$$

$$= \frac{1}{4}$$

$$Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{1}{4} - \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$$

Theoretical exercises

7.19 Assume X and Y are identically distributed but not necessarily independent.

$$Cov(X+Y, X-Y) = Cov(X,X) + Cov(X,-Y) + Cov(Y,X) + Cov(Y,-Y)$$

$$Cov(X,-Y) = -Cov(X,Y)$$
$$Cov(Y,X) = Cov(X,Y)$$
$$Cov(x,x) = Var(X)$$
$$Cov(Y,-Y) = -Cov(Y,Y) = -Var(Y)$$

$$Cov(X+Y, X-Y) = Var(X) - Var(Y) + Cov(X,Y) - Cov(X,Y)$$
$$= Var(X) - Var(Y)$$

Since X and Y have same distribution,

$$Var(X) = Var(Y)$$

so $Cov(X+Y, X-Y) = 0$

Ch. 8 problems:

8.1 Suppose X has mean 20 and variance 20

$$P\{0<X<40\} = P\{|X-20|<20\}$$

By Chebyshev, $P\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$

So $P\{|X-20| \geq 20\} \leq \frac{20}{(20)^2} = \frac{1}{20}$

so $P\{|X-20| < 20\} \geq 1 - \frac{1}{20} = \frac{19}{20}$

$$P\{0<X<40\} \geq \frac{19}{20}$$

8.4 $X_1, \ldots, X_{20}$ independent Poisson random variables with mean 1.

So $X = \sum_{i=1}^{20} X_i$ is Poisson with parameter $\lambda = 20$ $\quad \mu = 20$, $\sigma^2 = 20$

(a) $P\{X>15\} \leq \frac{E[X]}{15} = \frac{20}{15} = \frac{4}{3}$ by Markov.

(b) By Central limit theorem, X is approximately normal, as it is the sum of many small independent increments.

$P\{X > 15\} = P\{X > 15.5\}$ (continuity correction, as X is discrete)

$\mu = 20 \quad \sigma = \sqrt{20}$

$$P\{X > 15.5\} = P\left\{Z > \frac{15.5 - 20}{\sqrt{20}}\right\} = P\{Z > -1.00623\}$$

$$= P\{Z < 1.00623\} = \Phi(1.00623) = .8428$$

(computer)

8.5 50 numbers rounded and then added.

Let X_i = rounding error on ith number.

$$X = \text{total error} = \sum_{i=1}^{50} X_i .$$

Errors are independent and uniformly distributed on $(-.5, .5)$

$$\mu = E[X_i] = \frac{.5 + -.5}{2} = 0$$

$$\sigma^2 = Var(X_i) = \frac{.5 - (-.5)}{12} = \frac{1}{12}$$

Central limit theorem $\Rightarrow$

$$\frac{\sum_{i=1}^{50} X - 50\mu}{\sqrt{50}\,\sigma} = \frac{X}{\sqrt{50}\sqrt{1/12}}$$ is approximately standard normal

$$P\{|X|>3\} = P\left\{\left|\frac{X}{\sqrt{50/12}}\right| > \frac{3}{\sqrt{50/12}}\right\}$$

$$= P\{|Z| > 1.46969\} = 2(1-\Phi(1.46969))$$

$$= 0.1416 \quad \text{(computer)}$$

8.7 100 identical light bulbs

X_i = lifetime of ith bulb is exponential with $\mu = 5$ hours

$$5 = \mu = \frac{1}{\lambda} \Rightarrow \lambda = 1/5$$

$$\sigma^2 = \mathrm{Var}(X_i) = \frac{1}{\lambda^2} = 25, \quad \sigma = 5$$

$X = \sum_{i=1}^{100} X_i$ is total lifetime of bulbs

By CLT: $\dfrac{X - 100\mu}{\sigma\sqrt{100}} = \dfrac{X-500}{50}$ is approximately standard normal

$$P\{X > 525\} = P\left\{\frac{X-500}{50} > \frac{525-500}{50}\right\} = P\left\{Z > \frac{1}{2}\right\}$$

$$= 1 - P\left\{Z < \frac{1}{2}\right\} = 1 - \Phi\left(\frac{1}{2}\right) = .3085$$

Theoretical exercise

8.1 Show if X has mean μ and standard deviation σ

Then $P\{|X-\mu| \geq k\sigma\} \leq \dfrac{1}{k^2}$

Proof from Markov: $E[(X-\mu)^2] = \sigma^2$

$$P\{(X-\mu)^2 \geq k^2\sigma^2\} \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2}$$

But $(X-\mu)^2 \geq k^2\sigma^2 \iff |X-\mu| \geq k\sigma$

so $P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$

Proof from Chebyshev:

Chebyshev says $P\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$

set $\ell = \frac{k}{\sigma}$ so $k = \sigma\ell$

then we get $P\{|X-\mu| \geq \sigma\ell\} \leq \frac{\sigma^2}{(\sigma\ell)^2} = \frac{1}{\ell^2}$

Change dummy variable ℓ back to k to get desired form.