Conditional Probability

Office hours M 1-2, 3-4 W 9:30-10:30

$P(E|F)$ = probability that E occurs given that $\underbrace{F}_{\text{condition}}$ occurs

Importance (i) compute probability w/ partial information

(ii) break up problems into conditional ones which may be easier.

(iii) Reasoning about hypotheses/evidence (Bayes's Formula)

(iv) Can define "independent events".

Suppose we are dealt 2 cards from a 52 card deck:

[Cards 13 values 2 3 4 ⋯ 10 J Q K A

4 suits Spades, hearts, diamonds, clubs]

eg. A spades

Suppose dealt 2 cards $P(2 \text{ aces})$?

$$\binom{52}{2} = \text{total \# 2 card hands}$$

$$= \text{\# points in } S$$

$$\binom{4}{2} = \text{pairs of aces}$$

$$P(2 \text{ aces}) = \frac{\binom{4}{2}}{\binom{52}{2}} = \frac{\left(\frac{4 \cdot 3}{2}\right)}{\left(\frac{52 \cdot 51}{2}\right)} = \frac{4 \cdot 3}{52 \cdot 51}$$

Suppose get cards one at a time

$P(2 \text{ aces} \mid \text{first card is an Ace})$

Prob. of getting 2 aces given that first card is an ace.

Thinking of drawing second card as a new experiment.

3 aces out of 51 cards left

$$P(2 \text{ aces} \mid \text{first is Ace}) = \frac{3}{51}$$

$$P(1\text{st card is an Ace}) = \frac{4}{52}$$

See $P(2\text{ aces}) = P(\text{1st card is Ace}) \cdot P(2\text{ aces} \mid \text{1st card is A})$

$$\frac{4 \cdot 3}{52 \cdot 51} = \frac{4}{52} \cdot \frac{3}{51}$$

$$P(2\text{ aces} \mid \text{1st card is A}) = \frac{P(2\text{ aces})}{P(\text{1st card is Ace})}$$

Formalize

F = 1st card is A

E = 2nd card is A

"2aces" = EF

$$\circledast \quad P(E \mid F) = \frac{P(EF)}{P(F)}$$

We promote $\circledast$ to a definition

If E and F are events, we define $P(E|F)$ by $\circledast$

$P(E|F)$ = "Prob E given F"

"Prob E conditional on F"

Suppose x is an outcome. If F occurs then x is in F.

If we want E to also occur, we need x in E also

so ultimately x is in EF

That why we take $P(EF)$

$$1 = P(F|F) = \frac{P(FF)}{c} = \frac{P(F)}{c} \quad \text{so } c = P(F)$$

So $P(F)$ is a normalization.

Another interpretation: Given that F is known to occur: then we can replace the sample space S with the subset F.

(reduced sample space)

<u>Ex</u> Urn with r red and b blue balls

n Balls chosen in order w/o replacement

$(n \leq r+b)$

Suppose k of n chosen are blue

what is $P(\text{1st ball chosen is blue})$

We work in reduced samplespace

B_k = event that k blue balls are chosen.

Each of the outcomes in B_k is equally likely
(need to think)

Among the n balls chosen, the first is equally likely to be any of those n, and there are k chances for it to be blue so $P(\text{1st is blue} \mid B_k) = \frac{k}{n}$

Working w/ full sample space

B = first ball chosen is blue

B_k = k blue balls are chosen

$$P(B \mid B_k) = \frac{P(B B_k)}{P(B_k)}$$

$$P(B B_k) = P(B)\, P(B_k \mid B)$$

$$P(B \mid B_k) = \frac{P(B_k \mid B)\, P(B)}{P(B_k)}$$

Trick: reversing the order of the conditional probability

can do these parts combinatorially

$$P(B) = \frac{b}{r+b} \qquad P(B_k) = \frac{\binom{b}{k}\binom{r}{n-k}}{\binom{r+b}{n}}$$

$$P(B_k \mid B) = \frac{\binom{b-1}{k-1}\binom{r}{n-k}}{\binom{r+b-1}{n-1}}$$

$$P(B \mid B_k) = \frac{P(B_k \mid B) \cdot P(B)}{P(B_k)} = \frac{k}{n}$$