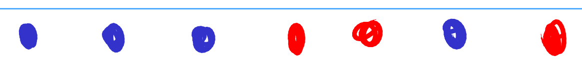
More examples with equally likely outcomes

Last time: Occupancy problem, Birthday Paradox

More about choice of sample space:

<u>Example</u> $n+m$ balls n red
m blue

Arrange the balls in row, randomly
what is the probability of getting
a particular sequence of colors?

$P($  $) = ?$ $\left(\begin{matrix} n=3 \\ m=4 \end{matrix}\right)$

$(n+m)!$ total orderings
groups of n indistinguishable
m indistinguishable objects

$\dfrac{(n+m)!}{n!\,m!}$ number of possible color sequence

If we regard the outcome of the experiment as being just the sequence of colors

Sample space has $\dfrac{(n+m)!}{n!\,m!}$ points

If each is equally likely,

$$P(\{\text{color sequence}\}) = \left[\frac{(n+m)!}{n!\,m!}\right]^{-1}$$

OR we could regard the experiment as taking one ball at a time and putting it in one of the $n+m$ possible positions.

$$\bullet_1\ \bullet_2\ \bullet_3\ \bullet_4 \cdots \bullet_n \qquad \bullet_1\ \bullet_2 \cdots \bullet_m$$

$S = \{\text{orderings of } n+m \text{ balls}\}$
has $(n+m)!$ points

Assume each is equally likely $\quad P(\{x\}) = \dfrac{1}{(n+m)!}$

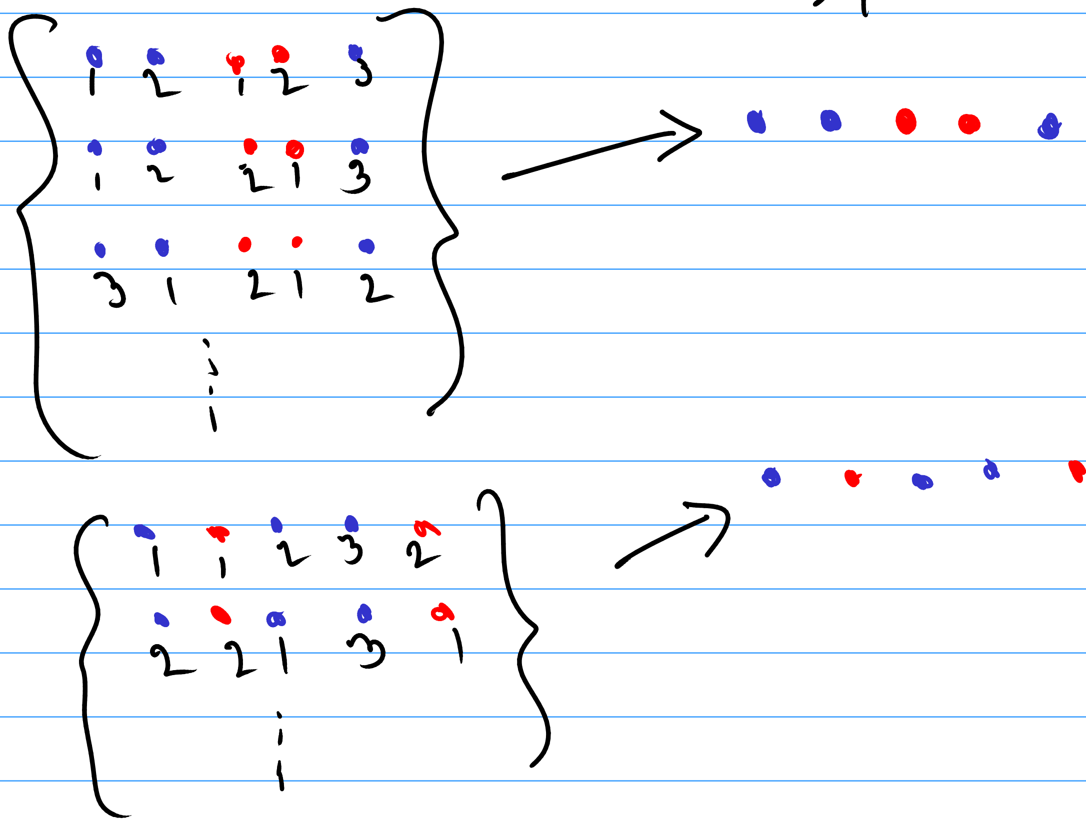
E = particular color sequence
consists of many points in sample space.

Given a particular color sequence, can order red balls in any way and blue balls in any way, to get a point in the sample space

$\#$ points in $E = n!\,m!$

$$P(E) = \frac{n!\,m!}{(n+m)!} = \left[\frac{(n+m)!}{n!\,m!}\right]^{-1}$$

$S_{w/order}$ $\qquad\qquad$ $S_{color\ sequences}$



$\#$ points going to each outcome in $S_{color\ sequences}$

is always the same namely $n!m!$

Example of inclusion-exclusion

Sports club N members

three sports tennis squash badminton

36 play tennis

28 squash

18 badminton

22 tennis & squash

12 tennis & badminton

9 squash & badminton

4 all three

So how many members play some racket sport?

Consider experiment where we randomly select a member of the club.

$S = \{ \text{members of the club} \}$

$P(\{\text{each member}\}) = \frac{1}{N}$

$T =$ set of members plays tennis

$P(T) = \frac{\#\text{ of members that play}}{N}$

$S =$ members plays squash

$B =$ " " badminton

At least one sport $\Leftrightarrow$ $T \cup S \cup B$

$P(T \cup S \cup B) = \frac{\#\text{ play at least one sport}}{N}$

$$P(T \cup S \cup B) = P(T) + P(S) + P(B)$$
$$- P(TS) - P(TB) - P(SB)$$
$$+ P(TSB)$$

$P(T) = \frac{36}{N}$ $P(S) = \frac{28}{N}$ $P(B) = \frac{18}{N}$

$P(TS) = \frac{22}{N}$ $P(TB) = \frac{12}{N}$ $P(SB) = \frac{9}{N}$

$P(TSB) = \frac{4}{N}$

$$\frac{36 + 28 + 18 - 22 - 12 - 9 + 4}{N} = \frac{\text{\# play at least one}}{N}$$

100 people go to a party put coats in a closet

when they leave, they grab a coat randomly

what is the probability that no person selects their own coat?

$= 1 - P(\text{at least one person selects own coat})$

E_i = person i selcets own coat

atleast one selects own $= \bigcup_{i=1}^{100} E_i$

$$P\left(\bigcup_{i=1}^{100} E_i\right) = \sum_{i=1}^{100} P(E_i) - \sum_{i<j} P(E_i E_j)$$

$$+ \sum_{i<j<k} P(E_i E_j E_k) - \text{4-way intersection} \cdots$$

$$P(E_i) = \frac{99!}{100!}$$ i gets own

$$P(E_i E_j) = \frac{98!}{100!}$$ i & j get own

$$P(E_{i_1} E_{i_2} \cdots E_{i_n}) = \frac{(100-n)!}{100!}$$ n people get own

$$P(E_{i_1} E_{i_2} \cdots E_{i_n}) = \frac{(100-n)!}{100!} \Big\} \text{ term like this appears } \binom{100}{n} \text{ times}$$

$$\binom{100}{n} \frac{(100-n)!}{100!} = \frac{100!}{(100-n)!\,n!} \frac{(100-n)!}{100!} = \frac{1}{n!}$$

$$P(\cup E_i) = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} \cdots - \frac{1}{100!}$$

$$1 - P(\cup E_i) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \cdots + \frac{1}{100!}$$

$$\approx 0.368$$

Compare $$\frac{1}{e} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!}$$

(The matching problem)