Equully Likely Outcomes.

Ch 2 problems: 8, 13, 18, 22, 25, 32, 39, 41, 54

Ch 2 theoretical: 10, 11, 18, 19

Ch 3 theoretical: 3.2

Today examples

Sample space $S = \{x_1, \dots, x_N\}$

$$P(\{x_1\}) = P(\{x_2\}) = \cdots = P(\{x_N\}) = \frac{1}{N}$$

In this situation, we say the sample space has equally likely outcomes.

For more general event $E \subset S$

$$P(E) = \frac{\#\text{outcomes in } E}{N} = \text{proportion of sample space in } E$$

So we can compute Probabilities just by counting.

Ex roll 2 dice. What is probability that the sum is 7?

$$S = \{ (1,1) \quad (1,2) \quad \cdots \quad (1,6)$$
$$(2,1) \quad (2,2) \quad \cdots \quad (2,6)$$
$$\vdots$$
$$(6,1) \quad \cdots \quad (6,6) \}$$

$N = 6 \cdot 6 = 36$

Assumption: each outcome is equally likely.

E = sum is 7 = $\{ (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) \}$

6 outcomes in E.

$$P(E) = \frac{6}{36} = \frac{1}{6}$$

Ex Group of 11 people 6 men, 5 women

3 people "randomly" select of the 11.

$P($1 man, 2 women$)$

S = set of $\binom{11}{3}$ subsets of the 11 people of size 3.

By "randomly selected" we mean each subset is equally likely.

$$\underset{\text{men}}{\binom{6}{1}}\underset{\text{women}}{\binom{5}{2}} = \text{\# of subsets with } 1 \text{ man, } 2 \text{ women}$$

$$P(1 \text{ man}, 2 \text{ women}) = \frac{\binom{6}{1}\binom{5}{2}}{\binom{11}{3}} = \frac{4}{11}$$

Or I could do: $S = \{\text{ordered selection of 3 people}\}$

$$N = \underset{\substack{\text{choices}\\ \text{for 1st}}}{11} \cdot \underset{\text{2nd}}{10} \cdot \underset{\text{3rd}}{9} = 990$$

$$\# \, M, W, W = 6 \cdot 5 \cdot 4 = 120$$

$$\# \, W, M, W = 5 \cdot 6 \cdot 4 = 120$$

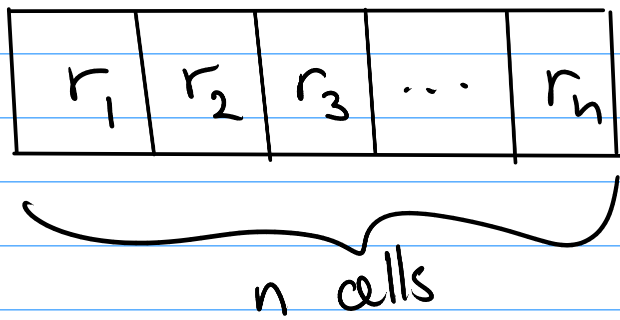
$$\# \, W, W, M = 5 \cdot 4 \cdot 6 = 120$$

$$P(1 \text{ man}, 2 \text{ women}) = \frac{120 + 120 + 120}{990} = \frac{4}{11}$$

ordered subsets equally likely $\Longleftrightarrow$ unordered subsets equally likely

$3! = 6$ ordered subsets $\longleftrightarrow$ 1 unordered subset

The Occupancy Problem : n cells, r particles/balls



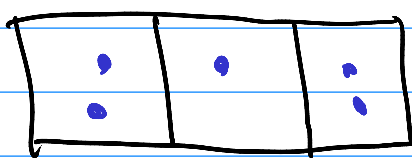
The particles are distributed "RANDOMLY", and we end up with some number in each cell.

r_i in the ith cell

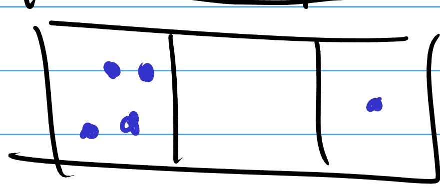
call $(r_1, r_2, \cdots, r_n)$ the "occupancy vector"

What is the probability of getting a particular occupancy vector ?

$n = 3$ cells $r = 5$ particles



$(r_1, r_2, r_3) = (2, 1, 2)$



$(r_1, r_2, r_3) = (4, 0, 1)$

Put each of r particles in one of n cells, randomly (each cell equally likely)

$\underbrace{n \cdot n \cdot n \cdots n}_{r} = n^r$ points in sample space

Assume each of these is equally likely $\frac{1}{n^r}$

How many ways to get $(r_1, r_2, \ldots, r_n)$?

$$\binom{r}{r_1}\binom{r-r_1}{r_2}\binom{r-r_1-r_2}{r_3}\cdots$$

$$= \frac{r!}{\cancel{(r-r_1)!}\,r_1!}\,\frac{\cancel{(r-r_1)}!}{(r-r_1-r_2)!\,(r_2)!}\cdots$$

$$= \frac{r!}{r_1!\,r_2!\cdots r_n!}$$

$$P((r_1, r_2, \ldots, r_n)) = \frac{r!}{r_1!\,r_2!\cdots r_n!}\,\frac{1}{n^r}$$

Particle which obey this probability law are said to have MAXWELL-BOLTZMANN statistics.

Note: actual elementary particles satisfy instead

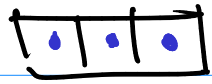
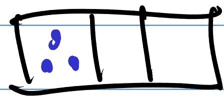
BOSE-EINSTEIN statistics (bosons) or FERMI-DIRAC statistics (Fermions)

Each occupancy vector is equally likely

$$\binom{r+n-1}{n-1}^{-1}$$

each $r_i = 0$ or 1 (Pauli exclusion principle)

$$\binom{n}{r}^{-1}$$

$n = r = 3$	Maxwell Boltzmann	Bose Einstein
	$\frac{2}{9}$	$\frac{1}{10}$
	$\frac{1}{27}$	$\frac{1}{10}$

Example (Birthday Paradox)

If n people are in a room, what is probability that 2 have the same birthday

(relation to previous: person $\leftrightarrow$ particle, birthday $\leftrightarrow$ cell)

Assume each person has each birthday with equal probability. (365 days No leap years)

$(365)^n$ points in sample space

How many ways to have no two people with same b-day?

$$\underbrace{(365)}_{\text{1st person}}\underbrace{(364)}_{\text{2nd person}}\cdots\underbrace{(365-n+1)}_{n\text{ th person}}$$

$$P(\text{no two have same}) = \frac{(365)\cdots(365-n+1)}{(365)^n}$$

$P(\text{two have same}) = 1 - P(\text{no two have same b-day})$

if $n \geq 23$

$P(\text{two have same}) > \frac{1}{2}$