Basic Properties, Inclusion-Exclusion

New OFFICE HOURS M 1-2, 3-4
W 9:30-10:30

S sample space
$E \subset S$ an event

$P(E)$ = probability of E

Axiom 1 $0 \leq P(E) \leq 1$

Axiom 2 $P(S) = 1$

Axiom 3 $E_1, E_2, \ldots, E_n, \ldots$ are mutually exclusive

$(E_i E_j = \emptyset \text{ unless } i = j)$

Then $P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$

Prop $P(\emptyset) = 0$ Did this last time.

Prop $P(E^c) = 1 - P(E)$

Proof:

$S = E \cup E^c$ Check Axiom 3 applies

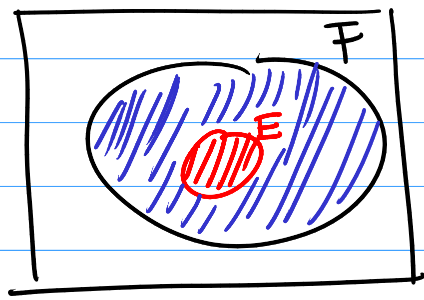
$EE^c = \emptyset$

$1 \overset{Ax2}{=} P(S) \overset{Ax3}{=} P(E) + P(E^c)$ QED.

Prop If $E \subset F$
(whenever E occurs, F also occurs)

THEN $P(E) \leq P(F)$

Proof:



$F = E \cup FE^c$

mutually exclusive

$(EFE^c = (EE^c)F = \emptyset F = \emptyset)$

Axiom 3 $P(F) = P(E) + P(FE^c)$

By axiom 1 $P(FE^c) \geq 0 \Rightarrow P(F) \geq P(E)$

QED

[E.g. rolling die $S = \{1, \dots, 6\}$
$P(\{2\}) \leq P(\{\text{even}\}) = P(\{2,4,6\})$

Inclusion–Exclusion Identity

→ compute probability of a union of events, when the events <u>are not</u> assumed to be mutually exclusive.

$P(E \cup F)$

Rolling 6-sided die

$P(\{\text{even}\}) = P(\{2,4,6\}) = \frac{1}{2}$

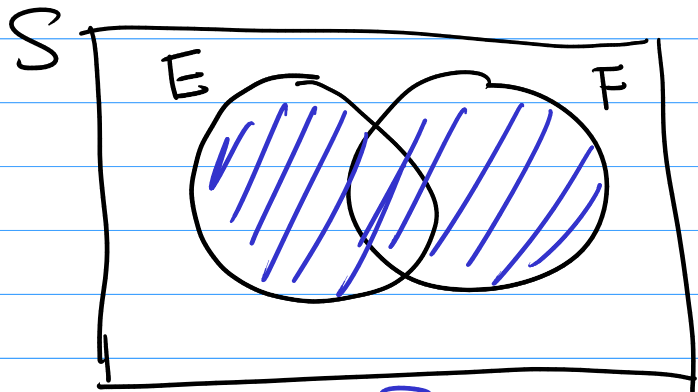
$P(\{\text{prime}\}) = P(\{2,3,5\}) = \frac{1}{2}$

$P(\{\text{even}\} \cup \{\text{prime}\}) = P(\{2,3,4,5,6\}) = \frac{5}{6}$

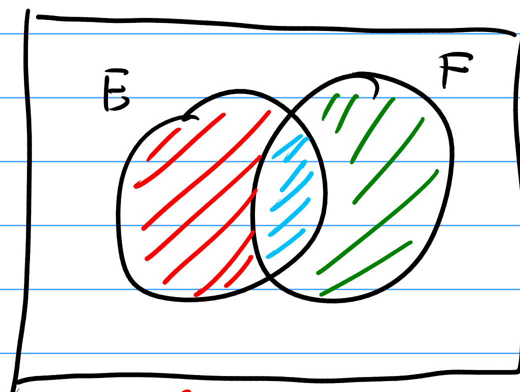
(inclusion-exclusion for 2 events)

<u>Prop</u> $P(E \cup F) = P(E) + P(F) - P(EF)$

Break up $E \cup F$ into mutually exclusive events



$E \cup F$



$EF^c \cup EF \cup FE^c$

$$P(E \cup F) \overset{\text{Axiom 3}}{=} P(EF^c) + P(EF) + P(FE^c)$$

$$\begin{cases} P(E) = P(EF^c) + P(EF) \\ P(F) = P(EF) + P(FE^c) \end{cases}$$

$$P(E) + P(F) = P(EF^c) + \underbrace{2P(EF)}_{\text{intersection is overcounted}} + P(FE^c)$$

$$P(E) + P(F) - P(EF) = P(EF^c) + P(EF) + P(FE^c) = P(E \cup F)$$

Example: I'm trying to find gifts my mom. I find two.
I estimate:

$P(L_1) = P(\{\text{she likes first gift}\}) = 0.5$

$P(L_2) = P(\{\text{she like second gift}\}) = 0.4$

$P(L_1L_2) = P(\{\text{she likes both}\}) = 0.3$

$P(\{\text{she like neither}\}) = P(L_1^cL_2^c)$

$P(L_1 \cup L_2) = P(L_1) + P(L_2) - P(L_1L_2)$

$= 0.6$

$P(L_1^cL_2^c) \underset{\uparrow}{=} P((L_1 \cup L_2)^c) = 1 - P(L_1 \cup L_2) = 0.4$

De Morgan's law

n-event inclusion exclusion:

$\binom{n}{2}$ pairwise intersections

$$P(E_1 \cup E_2 \cup \dots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1}E_{i_2})$$

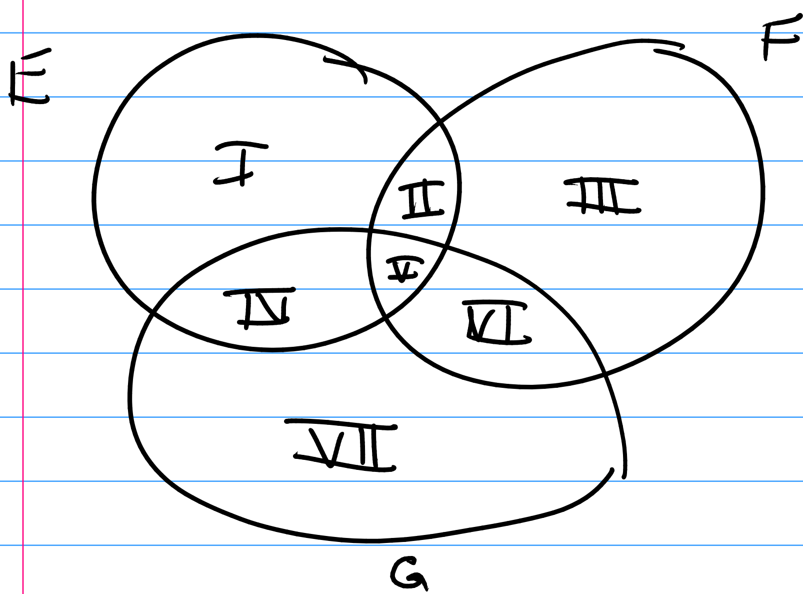
$$+ \sum_{i_1 < i_2 < i_3} P(E_{i_1}E_{i_2}E_{i_3})$$ $\binom{n}{3}$ triple intersections

$$- \sum_{i_1 < i_2 < i_3 < i_4} P(E_{i_1}E_{i_2}E_{i_3}E_{i_4})$$ $\binom{n}{4}$ quadruple intersections

cont →

$$+\cdots+(-1)^{n+1}P(E_1E_2\cdots E_n)$$

3 sets $P(E\cup F\cup G)$



$$P(E)+P(F)+P(G) = P(\mathrm{I})+P(\mathrm{III})+P(\mathrm{VII}) + 2P(\mathrm{II}) + 2(\mathrm{IV}) + 2P(\mathrm{VI}) + 3P(\mathrm{V})$$

$$-P(EF)-P(EG)-P(FG) = -P(\mathrm{II})-P(\mathrm{IV})-P(\mathrm{VI}) - 3P(\mathrm{V})$$

$$+P(EFG) = P(\mathrm{V})$$

Add it up $\Rightarrow$ get each region exactly once.