Set operations, Axioms of Probability

Recall from Last time:

S = sample space = set of possible outcomes

E an event = a subset of sample space

Union $E \cup F$ = "E or F or both"

Intersection EF = "E and F"

Empty set $\emptyset$ = set with no elements

it represents an event which is logically impossible

e.g. Flip coin get both Heads and Tails

$$\{H\} \cap \{T\} = \emptyset$$

Two events E and F are <u>mutually exclusive</u> if it is logically impossible for them to occur simultaneously

$$EF = \emptyset$$

Complement E^c = all outcomes in S that are not in E.

$S = \{H, T\}$

$E = \{H\}$ $\quad$ $E^c = \{T\}$

Notation

$$E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n$$

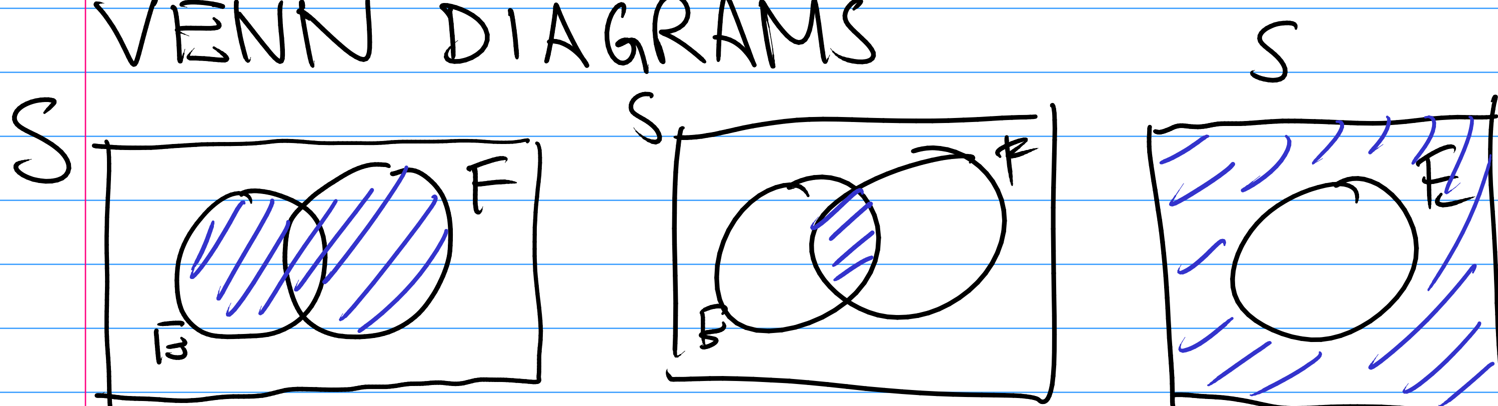
$$= \bigcup_{i=1}^{n} E_i$$

$$E_1 \cap E_2 \cap E_3 \cdots \cap E_n = \bigcap_{i=1}^{n} E_i$$

Also have $\displaystyle\bigcup_{i=1}^{\infty} E_i$ $\quad$ $\displaystyle\bigcap_{i=1}^{\infty} E_i$

(Analogous to Σ notation $\displaystyle\sum_{i=1}^{\infty} a_i$)

VENN DIAGRAMS



$E \cup F$ $\qquad$ EF $\qquad$ E^c

Laws for Set operations (Boolean Logic)

Commutative law $E \cup F = F \cup E$ $\quad EF = FE$

Associative $(E \cup F) \cup G = E \cup (F \cup G)$

$$(EF)G = E(FG)$$

Distribution law $(E \cup F)G = (EG) \cup (FG)$

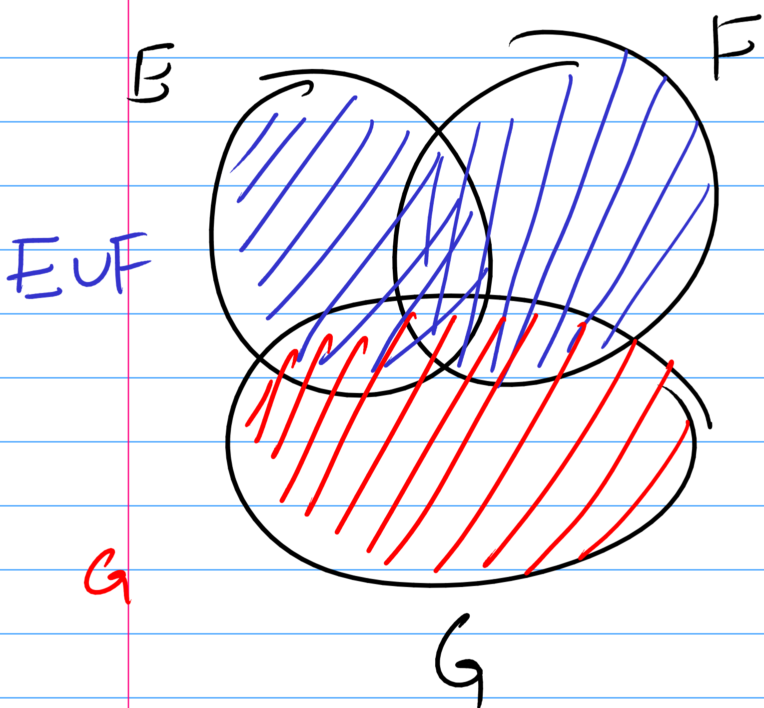
$$(EF) \cup G = (E \cup G)(F \cup G)$$

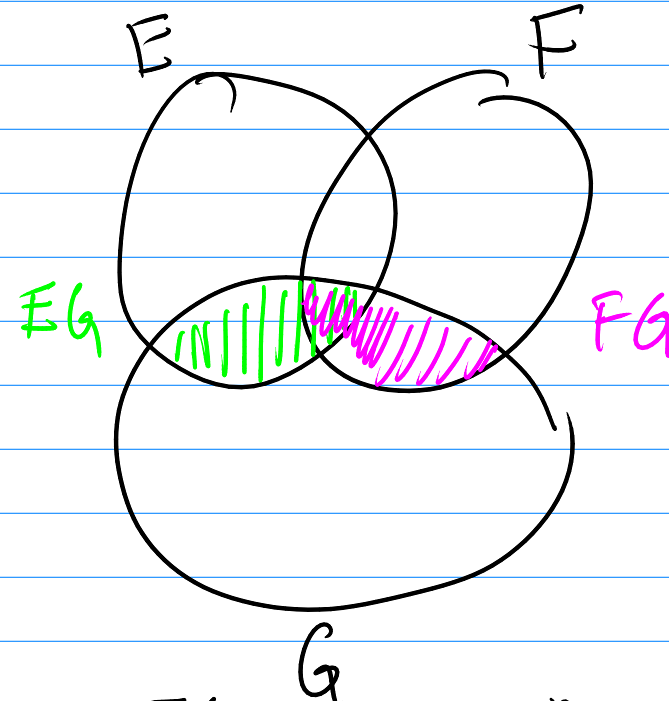
De Morgan's law

$$\begin{cases} (E \cup F)^c = E^c \cap F^c \\ \left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c \end{cases}$$

$$\begin{cases} (E \cap F)^c = E^c \cup F^c \\ \left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c \end{cases}$$

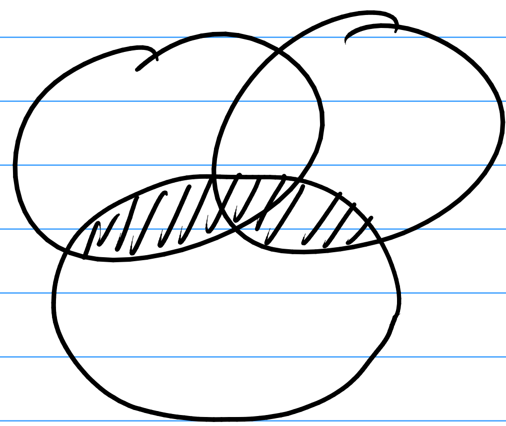
Prove $(E \cup F)G = (EG) \cup (FG)$





$(E \cup F)G$ = both red and blue

$EG \cup FG$ = either green or purple



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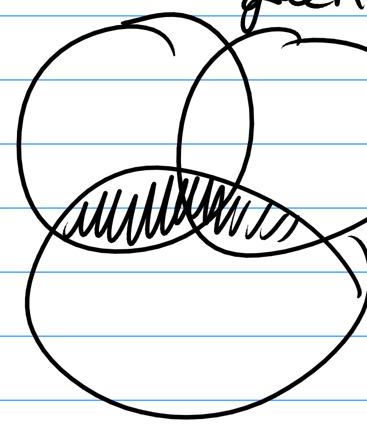
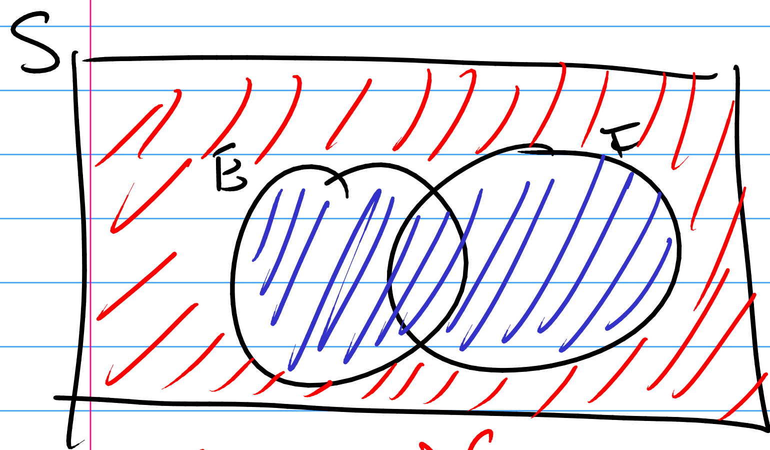
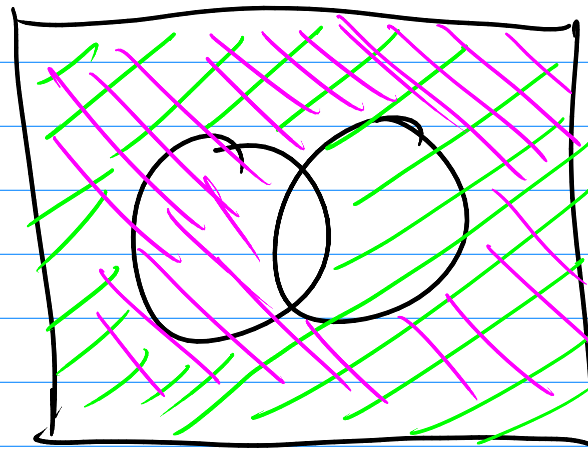


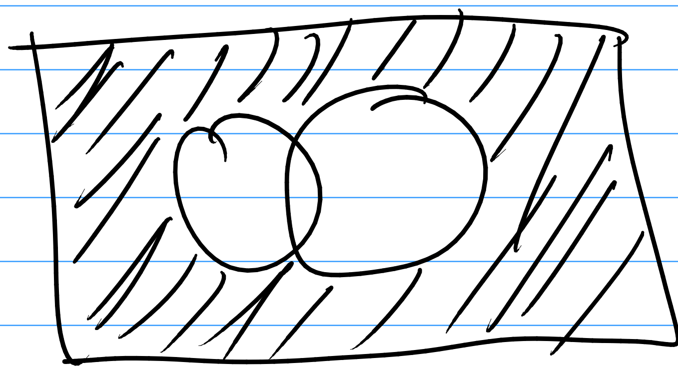
De Morgan's

$$(E \cup F)^c = E^c \cap F^c$$



$(E \cup F)^c$





Axioms of Probability

S = sample space

A measure of probability in S is an assignment of a number

$$P(E) = \text{"the probability of } E\text{"}$$

to each event E contained in the sample space.

<u>Axiom 1</u> $\quad 0 \leq P(E) \leq 1$

<u>Axiom 2</u> $\quad P(S) = 1$

<u>Axiom 3</u> If $E_1, E_2, E_3, \ldots$ is a sequence of events which are <u>mutually exclusive</u>

$$(\, E_i E_j = \emptyset \quad \text{unless } i = j \,)$$

Then $$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

"For mutually exclusive events, the prob of the union is the sum of the probabilities"

Remark 1 Axiom 3 applies to finite sequences of events

Remark 2 Axiom 3 asserts convergence of the infinite sum.

<u>Prop</u> $P(\emptyset) = 0$

Use axiom 3 $E_1 = S \quad E_2 = \emptyset$

* $P(S \cup \emptyset) = P(S) + P(\emptyset)$

(Check $S, \emptyset$ mutually exclusive

$S\emptyset = \emptyset \quad \checkmark$)

$S \cup \emptyset = S$ (like adding nothing)

$$P(S) = P(S) + P(\emptyset)$$

Axiom 2 $\quad P(S) = 1, \quad P(S) = 1$

$1 = 1 + P(\emptyset)$

$P(\emptyset) = 0 \quad \blacksquare$

Axiom 2 forces S to be nonempty

Examples

$S = \{H, T\}$

$P(\{H\}) = \frac{1}{2}$

$P(\{T\}) = \frac{1}{2}$

$P(\emptyset) = 0$ $\quad P(\{H,T\}) = 1$

Axioms 1, 2, 3 satisfied ✓

More generally: finite sample space

$$S = \{x_1, x_2, \ldots, x_n\}$$

Pick n numbers p_i, $i = 1, \ldots, n$ such that $0 \le p_i \le 1$ and $\sum_{i=1}^{n} p_i = 1$

Define: $P(\{x_i\}) = p_i$

$\boxed{\{x_i\} \text{ simple event} = \text{event with exactly one element}}$

Define $P(E) = \sum_{i \text{ such that } x_i \in E} p_i$

(Forced by Axiom 3)

This satisfies Axioms 1, 2, 3 ✓.

Infinite sample space

$S = \{0, 1, 2, \dots\} =$ non negative integers

$$P(\{i\}) = e^{-1} \frac{1}{i!} \qquad e = 2.71828\dots$$

E event

$$P(E) = \sum_{i \in E} e^{-1} \frac{1}{i!}$$

Check Axioms 1, 2, 3.