Final exam Friday May 11, 2-5pm, usual room

→ Two sheets of notes are permitted

→ Office hours
Monday May 7 1-4pm
Wednesday May 9 9:30-12

$X_1, X_2, \ldots, X_n, \ldots$ a sequence of independent random variables, all having the same distribution

Let $\mu = E[X_i]$ common value of mean.

Weak law of large numbers

$$\lim_{n\to\infty} P\left\{\left|\frac{X_1+\cdots+X_n}{n} - \mu\right| \geq \varepsilon\right\} = 0$$

One enhancement is the central limit theorem
→ gives the limiting/asymptotic shape of the distribution:

$\boxed{\text{Assume finite variance}}$

$Var(X_i) = \sigma^2$

$Var(\Sigma X_i) = n\sigma^2$

$$\frac{X_1+\cdots+X_n - n\mu}{\sigma\sqrt{n}}$$ is approximately normal with mean 0 and variance 1

Another enhancement of the weak law of large numbers is the strong law of large numbers, which involves a stronger notion of convergence.

What if we look at the sequence

$$A_n = \frac{X_1 + \cdots + X_n}{n}$$

Does this sequence have a limit $\lim_{n\to\infty} A_n$?

Strong law of large numbers says the limit exists and is equal to μ, with probability 1

("with probability 1" =: "almost surely")

X = uniform random variable on $(0,1)$

$$f_X(x) = \begin{cases} 1 & \text{if } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

$$P\{X = \tfrac{1}{2}\} = \int_{1/2}^{1/2} 1 \, dx = 0$$

Intuitively, "with probability 1" means "certainly"!

Strong law of large numbers.

$$P\left\{\lim_{n\to\infty} \frac{X_1+\cdots+X_n}{n} = \mu\right\} = 1$$

Eg.

n	1	2	3	4	5	
A_n	0	1	0	1	0	...

limit doesn't exist because sequence oscillates. But Strong law of large numbers says that this situation has probability zero.

$$P\left\{\lim_{n\to\infty} \frac{X_1+\cdots+X_n}{n} \text{ does not exist or exists but does not equal } \mu\right\} = 0$$

<u>Application</u> Let E be some event has probability $P(E)$

$$X_i = \begin{cases} 1 & \text{if } E \text{ occurs on the } i\text{th trial} \\ 0 & \text{if } E \text{ does not occur} \end{cases}$$

$$E[X_i] = P(X_i=1) = P(E)$$

With probability 1: $\displaystyle\lim_{n\to\infty} \frac{X_1+\cdots+X_n}{n} = P(E)$

Weak law $\lim_{n\to\infty} P\left\{\left|\frac{X_1+\cdots+X_n}{n}-\mu\right| \geq \varepsilon\right\} = 0$

Means if we take n sufficiently large

then the prob that $\left|\frac{X_1+\cdots X_n}{n}-\mu\right| \geq \varepsilon$ is small

But this leaves open the possibility that

$$\left|\frac{X_1+\cdots+X_n}{n}-\mu\right| \geq \varepsilon \text{ for infinitely many } n.$$

The strong law of large numbers rules this out:
with probability 1, there are only
finitely many values of n for which

$$\left|\frac{X_1+\cdots+X_n}{n}-\mu\right| \geq \varepsilon.$$

i.e., there is some n_0 such that for all

$$n \geq n_0, \quad \left|\frac{X_1+\cdots+X_n}{n}-\mu\right| < \varepsilon$$

Proof of Strong law of large numbers

We will assume that $E[X_i^4] = K < \infty$

finite forth moment. [The theorem is true without this assumption.]

Case 1: assume $\mu = E[X_i] = 0$.

$S_n = X_1 + \cdots + X_n$ sequence of sums

Want to show $\lim_{n\to\infty} \frac{S_n}{n} = 0$

$$E[S_n^4] = E[(X_1 + \cdots + X_n)^4]$$

= sum of terms like

$$E[X_i^4], E[X_i^3 X_j], E[X_i^2 X_j^2]$$

$i \neq j \neq k \neq l$

$$E[X_i^2 X_j X_k], E[X_i X_j X_k X_l]$$

Most terms go away

$$E[X_i^3 X_j] = E[X_i^3] E[X_j] \text{ by independence}$$

$$= E[X_i^3] \cdot 0 \quad b/c \ \mu = 0.$$

$$E[X_i^2 X_j X_k] = E[X_i^2] E[X_j] E[X_k] = 0$$

$E[X_i X_j X_k X_\ell] = 0$ similarly

Left with $E[X_i^4]$ and $E[X_i^2 X_j^2]$ $i \neq j$

$$\overset{\parallel}{E[X_i^2]E[X_j^2]}$$

$E[X_i^4]$ occurs once for each i

$E[X_i^2 X_j^2]$ occurs $\binom{4}{2} = 6$ times for each pair i, j

Assumed $E[X_i^4] = K < \infty$

so $0 \leq Var(X_i^2) = E[X_i^4] - (E[X_i^2])^2 = K - (E[X_i^2])^2$

$$E[X_i^2 X_j^2] = E[X_i^2]E[X_j^2] = (E[X_i^2])^2 \leq K$$

Thus

$$E[S_n^4] = \sum_{i=1}^n E[X_i^4] + 6 \sum_{i \neq j} E[X_i^2 X_j^2]$$

$$\leq nK + 6\binom{n}{2}K$$

$$= nK + 3n(n-1)K$$

$$\leq nK + 3n^2 K$$

$$E\left[\frac{S_n^4}{n^4}\right] \leq \frac{K}{n^3} + \frac{3K}{n^2}$$

$$E\left[\sum_{n=1}^{\infty} \frac{S_n^4}{n^4}\right] = \sum_{n=1}^{\infty} E\left[\frac{S_n^4}{n^4}\right] \leq \sum_{n=1}^{\infty} \frac{K}{n^3} + \frac{3K}{n^2} < \infty$$

converges by p-test

$E\left[\sum_{n=1}^{\infty} \frac{S_n^4}{n^4}\right] < \infty$ implies

that $$P\left\{ \sum_{n=1}^{\infty} \frac{S_n^4}{n^4} < \infty \right\} = 1$$

Finite expectation $\Rightarrow$ almost surely finite

$$\sum_{n=1}^{\infty} \frac{S_n^4}{n^4} < \infty \Rightarrow \lim_{n\to\infty} \frac{S_n^4}{n^4} = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{S_n}{n} = 0$$ by nth term test

$$P\left\{ \lim_{n\to\infty} \frac{S_n}{n} = 0 \right\} = 1$$

Case $\mu \neq 0$, just replace X_i by $X_i - \mu$