Exam Stats

$X = P1 + P2 + P3 + P4$ First 4 problems

$X' = 80 - \frac{1}{2}(80 - x)$ curve for first 4

$Y = \frac{100}{80} X'$ out of 100

$Z = Y + P5$

mean	86.1
1Q	75.6
2Q	94.25
3Q	96.2

Reminder: Course Instructor survey

Due: 5/4

<u>Last time</u> If X and Y are R.V.s.

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

$$E[X+Y] = E[X] + E[Y]$$ (does not require independence)

Lemma

If X and Y are independent then

$$E[g(X)h(Y)] = E[g(X)] \cdot E[h(Y)]$$

Pf. $$E[g(X)h(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f(x,y)\,dx\,dy$$

$$= \iint g(x)h(y)f_X(x)f_Y(y)\,dx\,dy$$ ← uses independence

$$= \int g(x)f_X(x)\left[\int h(y)f_Y(y)\,dy\right]dx$$

$$= \left[\int g(x)f_X(x)\,dx\right]\left[\int h(y)f_Y(y)\,dy\right]$$

$$= E[g(X)] \cdot E[h(Y)]$$

Cor $E[XY] = E[X]E[Y]$

PROVIDED that X and Y are independent

Let X and Y be random variable, not assumed to be independent:

Notation $\mu_X = E[X]$ $\mu_Y = E[Y]$

$\sigma_X^2 = Var(X)$ $\sigma_Y^2 = Var(Y)$

<u>Def</u> Covariance:

$$\mathrm{Cov}(X,Y) := E[(X-\mu_X)(Y-\mu_Y)]$$
$$= E[(X-E[X])(Y-E[Y])]$$

<u>Prop</u>: $\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$

$$E[(X-\mu_X)(Y-\mu_Y)] = E[XY - \mu_X Y - \mu_Y X + \mu_X\mu_Y]$$
$$= E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X\mu_Y$$
$$= E[XY] - \mu_X\mu_Y - \mu_Y\mu_X + \mu_X\mu_Y$$
$$= E[XY] - \mu_X\mu_Y = E[XY] - E[X]E[Y]$$

<u>Properties</u> (I) X, Y independent $\Rightarrow \mathrm{Cov}(X,Y) = 0$

(I') $\mathrm{Cov}(X,Y) \neq 0 \Rightarrow X$ and Y are dependent.

(II) $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$ symmetric

(III) $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$

(IV) $\mathrm{Cov}(aX,Y) = a\,\mathrm{Cov}(X,Y)$

(V) $\mathrm{Cov}(X_1+X_2, Y) = \mathrm{Cov}(X_1,Y) + \mathrm{Cov}(X_2,Y)$

(V') $\operatorname{Cov}\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{m} Y_j\right)$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \operatorname{Cov}(X_i, Y_j)$$

Variance of a sum of random variables

$$\operatorname{Var}(X_1+X_2) = \operatorname{Cov}(X_1+X_2, X_1+X_2)$$

$$= \operatorname{Cov}(X_1, X_1+X_2) + \operatorname{Cov}(X_2, X_1+X_2)$$

$$= \operatorname{Cov}(X_1,X_1) + \operatorname{Cov}(X_1,X_2) + \operatorname{Cov}(X_2,X_1) + \operatorname{Cov}(X_2,X_2)$$

$$= \operatorname{Cov}(X_1,X_1) + \operatorname{Cov}(X_2,X_2) + 2\operatorname{Cov}(X_1,X_2)$$

$$= \operatorname{Var}(X_1) + \operatorname{Var}(X_2) + 2\operatorname{Cov}(X_1,X_2)$$

Summarize $\operatorname{Var}(X_1+X_2) = \operatorname{Var}(X_1) + \operatorname{Var}(X_2) + 2\operatorname{Cov}(X_1,X_2)$

In general

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{Var}(X_i) + 2\sum_{j=2}^{n}\sum_{i=1}^{j-1} \operatorname{Cov}(X_i, X_j)$$

Cor If $X_1, X_2, \dots, X_n$ are independent

$$\operatorname{Cov}(X_i,X_j) = 0 \Rightarrow \operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{Var}(X_i)$$

Caution

$\mathrm{Cov}(X,Y)=0$ does not necessarily imply that X and Y are independent

Ex X has values $-1, 0, 1$ each w/ probability $\frac{1}{3}$

$$E[X] = -\frac{1}{3} + 0\frac{1}{3} + 1\cdot\frac{1}{3} = 0$$

$$Y = \begin{cases} 0 & \text{if } X \neq 0 \\ 1 & \text{if } X = 0 \end{cases}$$

So X and Y are not independent

BUT $XY \equiv 0$ $\quad E[XY] - E[X]E[Y] = 0 - 0\,E[Y] = 0$

Just mention correllation of X and Y

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

FACT $-1 \le \rho(X,Y) \le 1$

$\rho(X,Y) = 1 \Rightarrow$ perfect linear relationship between X and Y

$$Y = aX + b \quad \text{with} \quad a > 0$$

$\rho(X,Y) = -1 \Rightarrow$ perfect linear relationship $Y = aX + b$ with $a < 0$.